* UNIT-II *

Ordinary Differential Equations of Higher order :-

(1) Linear Differential Eq^ns of Second and Higher order :-

Definition:- An eq^n of the form $\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n(x)\cdot y = Q(x)$

where $P_1(x), P_2(x), P_3(x), \ldots, P_n(x)$ and $Q(x)$ (functions of x) continuous is called a linear D.E of order 'n'.

(2). Linear Differential Eq^ns with constant coefficients :-

Def:- An Eq^n of the form $\frac{d^n y}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + P_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n y = Q(x)$, where $P_1, P_2, \ldots, P_n$ are real constants and $Q(x)$ is a continuous function of 'x' is called an linear Differential Eq^n of order 'n' with constant coefficients.

Note:- (1) operator $D = \frac{d}{dx}$; $D^2 = \frac{d^2}{dx^2}$, ---- ; $D^n = \frac{d^n}{dx^n}$

$Dy = \frac{dy}{dx}$; $D^2y = \frac{d^2y}{dx^2}$; ---- ; $D^ny = \frac{d^ny}{dx^n}$

(2). operator $\frac{1}{D}Q = \int Q$ i.e, $D^{-1}Q$ is called the Integral of 'Q'.

Homogenous linear D.Eq's with constant coefficients :-

The general form of the Homogenous linear ②.

D.E of 2^{nd} order is $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0$ ——— (1).

where a, b, c are constants.

Put $\frac{d}{dx}=D$; $\frac{d^2}{dx^2}=D^2$ in (1) $\Rightarrow \left(a\frac{d^2}{dx^2}+b\frac{d}{dx}+c\right)y=0$

$\Rightarrow (aD^2+bD+c)y=0$.

(OR)

$f(D)y=0$.

To find the G.S of $f(D)y=0$:-

(i). write the A.E is $f(m)=0$

i.e, $am^2+bm+c=0$.

The above Eqn is called as A.E. Since it is a Q.E, we are solving these Eq's, we get the different types of roots. Here many cases will arise.

(ii). Depending on the nature of the roots, we write the complementary function. this is called G.S of given Homogenous linear D.Eq's with constant coefficients.

Consider the following table :-

S.No	Roots of A.E $\boxed{f(m)=0}$	Complementary function (C.F)
1	$m_1, m_2 \dots m_n$ are real & distinct	$y_c = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}$
2	$m_1, m_2, \dots m_n$ are roots and two roots are Equal. i.e, m_1, m_2 are equal and real (i.e (repeated twice) & the rest are real and different.	$y_c = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$
3	$m_1, m_2, \dots, m_n$ are real and three roots are equal i.e m_1, m_2, m_3 are Equal and real (i.e (repeated twice) & the rest are real and different.	$y_c = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \dots + c_n e^{m_n x}$
4	Two roots of A.E are Complex say $\alpha + i\beta$, $\alpha - i\beta$ and rest are real and distinct.	$y_c = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$
5	If $\alpha \pm i\beta$ are repeated twice and rest are real & distinct	$y_c = e^{\alpha x}\left[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x\right] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$

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⑥. If $\alpha \pm i\beta$ are repeated thrice and rest are real and distinct.

$$y_c = e^{\alpha x}\left[(c_1+c_2x+c_3x^2)\cos\beta x + (c_4+c_5x+c_6x^2)\sin\beta x\right] + c_7e^{m_7x} + \cdots + c_ne^{m_nx}.$$

⑦. If roots of A.E irrational say $\alpha \pm \sqrt{\beta}$ and rest are real and distinct.

$$y_c = e^{\alpha x}\left[c_1\cosh\sqrt{\beta}x + c_2\sinh\sqrt{\beta}x\right] + c_3e^{m_3x} + \cdots + c_ne^{m_nx}$$

Problems :-

① $\dfrac{d^2y}{dx^2} - 18\dfrac{dy}{dx} + 77y = 0.$

Sol:- The given Eqn can be written as $(D^2-18D+77)y = 0$

$f(D)y = 0.$

$\therefore$ A.E is $m^2 - 18m + 77 = 0$

$\Rightarrow m^2 - 11m - 7m + 77 = 0$

$\Rightarrow m(m-11) - 7(m-11) = 0$

$\Rightarrow (m-11)(m-7) = 0$

$\Rightarrow \boxed{m_1 = 11}; \boxed{m_2 = 7}$

Since the roots are real and distinct, then G.S is

$$\boxed{y = c_1e^{m_1x} + c_2e^{m_2x}}$$

$y = c_1e^{11x} + c_2e^{7x}$, where c_1, c_2 are constants.

(2). solve $\frac{d^3y}{dx^3}+\frac{d^2y}{dx^2}+4\frac{dy}{dx}+4y=0$ (or) $(D^3+D^2+4D+4)y=0$.

sol:- A.E is $m^3+m^2+4m+4=0$.

Put $m=-1 \Rightarrow -1+1-4+4=0$.

$0=0$.

$$\begin{array}{c|cccc} m=-1 & 1 & 1 & 4 & 4 \\ & 0 & -1 & 0 & -4 \\ \hline & 1 & 0 & 4 & \boxed{0} \end{array}$$

$\Rightarrow (m+1)(m^2+0m+4)=0$

$\Rightarrow (m+1)(m^2+4)=0$

$\Rightarrow \boxed{m_1=-1}$; $\boxed{m^2=-4} \Rightarrow \boxed{m=\pm 2i}$ $(m=\alpha\pm i\beta)$ Here $\alpha=0$, $\beta=2$.

$\therefore y= c_1e^{-x}+e^{0x}[c_1\cos 2x+c_2\sin 2x]$ $\left[\because y=c_1e^{m_1x}+e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)\right]$

$y=c_1e^{-x}+c_1\cos 2x+c_2\sin 2x$ //

(3). Solve $(D^4-D^3-9D^2-11D-4)y=0$ & (4). $(D^4+4D^3-5D^2-36D-36)y=0$

(4). sol: A.E : $m^4+4m^3-5m^2-36m-36=0$

Put $m=1 \Rightarrow \neq 0$

$m=-1 \Rightarrow \neq 0$

$m=2 \Rightarrow \neq 0$

$m=-2 \Rightarrow 16-32-20+72-36=0$

$0=0$.

$$\begin{array}{l|ccccc} m=-2 & 1 & 4 & -5 & -36 & -36 \\ & 0 & -2 & -4 & 18 & 36 \\ \hline m=-2 & 1 & 2 & -9 & -18 & \boxed{0} \\ & 0 & -2 & 0 & 18 & \\ \hline & 1 & 0 & -9 & \boxed{0} & \end{array}$$

$(m+2)(m+2)(m^2-9)=0$

$\Rightarrow m_1=-2,\ m_2=-2 \quad ; \quad m=\pm 3.\ (m_3=3\ ;\ m_4=-3)$

$\boxed{m_1=m_2=m}$

$\therefore y=(c_1+c_2x)e^{mx}+c_3e^{m_3x}+c_4e^{m_4x}$

$\Rightarrow y=(c_1+c_2x)e^{-2x}+c_3e^{3x}+c_4e^{-3x}$ //

4. $(D^3+16D)y=0.$

$f(D)y=0$, where $f(D)=D^3+16D$

AE is $f(m)=0$

$\Rightarrow m^3+16m=0$

$\Rightarrow m(m^2+16)=0$

$\Rightarrow \boxed{m_1=0}\ ; \boxed{m=\pm 4i}$

$\therefore y_c=c_1e^{m_1x}+e^{\alpha x}(c_2\cos\beta x+c_3\sin\beta x)$

$y_c=c_1e^{0x}+e^{0x}(c_2\cos 4x+c_3\sin 4x)$

$y_c=c_2\cos 4x+c_3\sin 4x+c_1$ //

(1). Find $\frac{1}{D} e^{2x}$

Sol^n:- $\frac{1}{D} e^{2x} = \int e^{2x} + c = \frac{e^{2x}}{2} + c.$

(2). $\frac{1}{D}(\sin 3x) = \int \sin 3x + c = \frac{-\cos 3x}{3} + c.$

(3). $\frac{1}{D^2}(\sin 3x) = \frac{1}{D}\left(\frac{1}{D} \sin 3x\right) = \frac{1}{D}\left[\left(\frac{-\cos 3x}{3} + c\right)\right] = \int\left(\frac{-\cos 3x}{3} + c\right)$

$= \frac{-\sin 3x}{9} + c.$

(4). $\frac{1}{D^2}(x^2+2) = \frac{1}{D}\left[\frac{1}{D}(x^2+2)\right] = \frac{1}{D}\left(\frac{x^3}{3} + 2x\right) = \frac{x^4}{12} + x^2 + c$ //

formulae :-

— * $\frac{1}{D+a}[f(x)] = e^{-ax} \int e^{ax} f(x)\, dx + c$

— * $\frac{1}{D-a}[f(x)] = e^{ax} \int e^{-ax} f(x)\, dx + c.$

(1) Find $\frac{1}{D-2} e^{2x} = e^{2x} \int e^{-2x} e^{2x}\, dx + c$

$= e^{2x} \int 1\, dx + c$

$= x e^{2x} + c$

(2) find $\frac{1}{D+2} e^{4x} = e^{-2x} \int e^{2x} e^{4x}\, dx + c$

$= e^{-2x} \int e^{6x}\, dx + c$

$= e^{-2x}\left(\frac{e^{6x}}{6}\right) + c$

$= \frac{e^{4x}}{6} + c$ //

③. Find $\frac{1}{D(D-2)}\sin 3x = \frac{1}{D-2}\left[\int \sin 3x\,dx\right]$ (10)

$$= \frac{1}{D-2}\left(\frac{-\cos 3x}{3}\right).$$

$$= \frac{-1}{3} e^{2x}\int e^{-2x}\cos 3x\,dx + c$$

$$= \frac{-e^{2x}}{3}\,\frac{e^{-2x}}{4+9}\left(-2\cos 3x + 3\sin 3x\right) + c$$

$$\left[\begin{array}{l} \because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\cos bx + b\sin bx\right) + c \\ \because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left(a\sin bx - b\cos bx\right) + c \end{array}\right]$$

$$\therefore \frac{1}{D(D-2)}\sin 3x = \frac{+1}{39}\left(2\cos 3x - 3\sin 3x\right) + c$$ 4

④. Find $\frac{1}{(D-1)(D-2)}e^{3x}$.

Sol: $\frac{1}{(D-1)(D-2)}e^{3x} = \frac{1}{D-1}\left(\frac{1}{D-2}e^{3x}\right)$

$$= \frac{1}{D-1}\left[e^{2x}\int e^{-2x}.e^{3x}\,dx\right]$$

$$= \frac{1}{D-1}\left[e^{2x}\int e^{x}\,dx\right]$$

$$= \frac{1}{D-1}\left(e^{2x}e^{x}\right)$$

$$= \frac{1}{D-1}\left(e^{3x}\right)$$

$$= e^{x}\int e^{-x}e^{3x}\,dx = e^{x}\int e^{2x}\,dx = e^{x}\frac{e^{2x}}{2} + c$$

$$= \frac{e^{3x}}{2} + c$$ 4

5. $\frac{1}{(D-2)(D+2)}\sin 3x$.

P.I $= \frac{1}{D-2}\left[\frac{1}{D+2}\sin 3x\right] = \frac{1}{D-2}\left[e^{-2x}\int e^{2x}\sin 3x\,dx\right]$

$$= \frac{1}{D-2}\left[e^{-2x}\left\{\frac{e^{2x}}{4+9}(2\sin 3x - 3\cos 3x)\right\}\right]$$

$$= \frac{1}{D-2}\left[e^{-2x}\frac{e^{2x}}{13}(2\sin 3x - 3\cos 3x)\right]$$

$$= \frac{1}{D-2}\frac{1}{13}(2\sin 3x - 3\cos 3x)$$

$$= \frac{1}{13}\left[\frac{1}{D-2}(2\sin 3x - 3\cos 3x)\right]$$

$$= \frac{1}{13}\left[\frac{1}{D-2}2\sin 3x - \frac{1}{D-2}3\cos 3x\right]$$

$$= \frac{1}{13}\left[\left(e^{2x}\int e^{-2x}2\sin 3x\,dx\right) - \left(e^{2x}\int e^{-2x}3\cos 3x\,dx\right)\right] + c$$

$$= \frac{1}{13}\left[2e^{2x}\frac{e^{-2x}}{4+9}(-2\sin 3x - 3\cos 3x)\right] - \frac{3}{13}\left[e^{2x}\frac{e^{-2x}}{4+9}(-2\cos 3x + 3\sin 3x)\right] + c$$

$$= \frac{2}{13\times 13}(-2\sin 3x - 3\cos 3x) - \frac{3}{13\times 13}(-2\cos 3x + 3\sin 3x) + c$$

$$= \frac{-4}{169}\sin 3x - \cancel{\frac{6}{169}\cos 3x} + \cancel{\frac{6}{169}\cos 3x} - \frac{9}{169}\sin 3x + c$$

$$= -\frac{13}{169}\sin 3x + c$$

$$= -\frac{1}{13}\sin 3x + c$$

6. P.I $\frac{1}{(D^2+3D+2)}(e^{4x}) = \frac{e^{4x}}{30}$.

$$\Rightarrow \frac{e^{4x}}{30}(D^2+3D+2) = \frac{16e^{4x}}{30} + \frac{12e^{4x}}{30} + \frac{2e^{4x}}{30}$$

$$= \frac{\cancel{30}e^{4x}}{\cancel{30}} = e^{4x}$$

Non Homogeneous Linear Differential Eq's with constant Coefficients :— (12)

The General form of the Non-Homogeneous Linear D.E of 2^{nd} order is.

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=Q(x)$$

$$\Rightarrow (aD^2+bD+c)\,y=Q(x)$$

$$\Rightarrow \boxed{f(D)y=Q(x)}$$

The solution of the above Eq^n is $\boxed{y=C.F+P.I}$

is $y=$ complementary function + particular Integral.

$$\boxed{\text{where } P.I=\frac{1}{f(D)}Q(x)}$$

Here $f(D)y=Q(x)$ contains no arbitrary constants.

finding particular Integrals in certain cases :—

case(1) :— P.I of $f(D)y=Q(x)$, when $\boxed{Q(x)=e^{ax}}$, where 'a' is constant. :—

If $Q(x)=e^{ax}$ then $P.I=\frac{1}{f(D)}Q(x)$

$$P.I=\frac{e^{ax}}{f(D)}$$

$$P.I=\frac{e^{ax}}{f(a)}\quad(\text{Put } D=a)$$

$$C.F = y_c = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$$

$$\therefore y_c = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x)$$

P.I :- $y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^2+4D+13} 2e^{-x}$

$$= \frac{2}{(-1)^2+4(-1)+13} e^{-x} \quad \left(\begin{array}{c} a=-1 \\ \because D=a \\ D=-1 \end{array} \right)$$

$$= \frac{2e^{-x}}{1-4+13}$$

$$= \frac{2e^{-x}}{10} = \frac{e^{-x}}{5}$$

$\therefore$ G.S is $y = y_c + y_p$

$$= e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{e^{-x}}{5}$$

(3). Solve the D.E $(D^2+2D+1)y = e^{-x}$.

Sol: y_c :- A.E is $f(m) = 0$

$$m^2+2m+1=0 \Rightarrow m = -1, -1.$$

$$\therefore C.F = y_c = (c_1+c_2x)e^{mx}$$

$$\boxed{y_c = (c_1+c_2x)e^{-x}}$$

y_p :- $y_p = \frac{1}{f(D)} \phi(x) = \frac{1}{(D^2+2D+1)} e^{-x} = \frac{e^{-x}}{(D+1)^2} = \frac{e^{-x}.x^2}{2!}$

$$\left(\begin{array}{c} a=-1 \\ \because D=a \\ D=-1 \end{array} \right)$$

$\therefore y = y_c + y_p$

$$= (c_1+c_2x)e^{-x} + \frac{e^{-x}.x^2}{2!}$$

4. Find the P.I of $(D^3+1)y = e^{-x}$

$$P.I = \frac{1}{f(D)}\phi(x) = \frac{1}{D^3+1}e^{-x} = \frac{e^{-x}}{(D+1)(D^2-D+1)} = \frac{e^{-x}}{(1+1+1)(D+1)^1} = \frac{e^{-x}}{3}\frac{x^1}{1!}$$

$$\begin{pmatrix} a=-1 \\ \theta=a \\ \theta=-1 \end{pmatrix}$$

$$= \frac{xe^{-x}}{3}$$

5. Solve $(4D^2-4D+1)y = 100$.

Sol: y_c:–

A.E is $f(m)=0$

$$4m^2-4m+1=0$$

$$\Rightarrow 4m^2-2m-2m+1=0$$

$$\Rightarrow 2m(2m-1)-(2m-1)=0$$

$$(2m-1)(2m-1)=0$$

$$m = 1/2, 1/2.$$

$$y_c = (c_1+c_2x)e^{+1/2 x}.$$

$$y_c = (c_1+c_2x)e^{x/2}.$$

y_p:– $y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{4D^2-4D+1}(100) = \frac{e^{0x}(100)}{4D^2-4D+1} = \frac{100}{1}$

$$= 100.$$

$$\begin{pmatrix} a=0 \\ \theta=a \\ \theta=0 \end{pmatrix}$$

$\therefore$ $y = y_c + y_p$

$$y = (c_1+c_2x)e^{x/2}+100.$$

6. Solve $(D^3-5D^2+8D-4)y = e^{2x}$

Ans: $\left(y = c_1e^{x}+(c_2+c_3x)e^{2x}+\frac{x^2e^{2x}}{2}\right).$

7. Solve the D.E $(D^3-6D^2+11D-6)y = e^{-2x}+e^{-3x}$.

Sol A.E is $f(m)=0$

$$\Rightarrow m^3-6m^2+11m-6=0$$

$$\Rightarrow (m-1)(m-2)(m-3)=0$$

$$\Rightarrow m_1=1,\ m_2=2,\ m_3=3.$$

$$\therefore y_c = c_1e^{m_1x}+c_2e^{m_2x}+c_3e^{m_3x}$$

$$\boxed{y_c = c_1e^{x}+c_2e^{2x}+c_3e^{3x}}$$

$$y_p = \frac{1}{(D^3-6D^2+11D-6)}\left(e^{-2x}+e^{-3x}\right)$$

$$= \frac{e^{-2x}}{D^3-6D^2+11D-6}+\frac{e^{-3x}}{D^3-6D^2+11D-6}$$

$$= \frac{e^{-2x}}{(-2)^3-6(-2)^2+11(-2)+6}+\frac{e^{-3x}}{(-3)^3-6(-3)^2+11(-3)-6}$$

$$= \frac{e^{-2x}}{-60}+\frac{e^{-3x}}{-120}$$

$$y_p = \frac{e^{-2x}}{-60}-\frac{e^{-3x}}{120}.$$

$$\boxed{\therefore y = y_c+y_p}.$$

8. $(D^2-3D+2)y = \cosh x$.

Sol A.E is $f(m)=0 \Rightarrow m^2-3m+2=0 \Rightarrow m_1=1, 2=m_2$

$$\therefore y_c = c_1e^{1x}+c_2e^{2x}$$

(18)

$$P.I = y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{(D^2-3D+2)} \cosh x = \frac{1}{D^2-3D+2}\left(\frac{e^x+e^{-x}}{2}\right)$$

$$= \frac{e^x}{2(D^2-3D+2)} + \frac{e^{-x}}{2(D^2-3D+2)}$$

~~Here~~ $\begin{pmatrix} a=1 \\ D=a \\ D=1 \end{pmatrix}$ $\begin{pmatrix} a=-1 \\ D=a \\ D=-1 \end{pmatrix}$

$$= \frac{e^x}{2(1-3+2)} + \frac{e^{-x}}{2(1+3+2)}$$

$$= \frac{e^x}{2(D-1)(D-2)} + \frac{e^{-x}}{12}$$

$$= \frac{e^x}{2(1-2)(D-1)} + \frac{e^{-x}}{12}$$

$$= \frac{e^x}{-2}\frac{x^1}{1!} + \frac{e^{-x}}{12}$$

$$\therefore y = y_c + y_p = c_1e^x + c_2e^{2x} + \frac{e^{-x}}{12} - \frac{xe^x}{2}$$

where c_1, c_2 are constants.

(9). Solve the D.E $(D^3-1)y = (e^x+1)^2$.

Sol: A.E is $m^3-1=0$.

$$(m-1)(m^2+m-1)=0.$$

$$m=1;\ m = \frac{-1\pm i\sqrt{3}}{2}.$$

$$y_c = c_1e^x + e^{-x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$$

$$y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^3-1}\left(e^x+1\right)^2$$

$$= \frac{1}{D^3-1}\left(e^{2x}+1+2e^x\right)$$

$$= \frac{1}{D^3-1}\left(e^{2x}\right) + \frac{1}{D^3-1} + \frac{2e^x}{D^3-1}$$

$$= \frac{1}{7}e^{2x} + \frac{1}{(-1)} + \frac{2e^x}{(D-1)(D^2+D+1)}$$

$$= \frac{1}{7}e^{2x} + \frac{1}{(-1)} + \frac{2e^x}{3(D-1)^1}$$

$$= \frac{1}{7}e^{2x} - 1 + \frac{2}{3}e^x \cdot \frac{x^1}{1!}$$

$$\therefore y = y_c + y_p$$

$$= c_1 e^x + e^{-x/2}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{7}e^{2x} + \frac{2xe^x}{3} - 1$$

(10). Solve $y'' - 4y' + 3y = 4e^{3x}$; $y(0) = -1$; $y'(0) = 3$.

Sol: Given Eqⁿ is $y'' - 4y' + 3y = 4e^{3x}$

i.e. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 4e^{3x}$

It can be expressed as.

$$(D^2 - 4D + 3)y = 4e^{3x}$$

$$\Rightarrow f(D)y = Q(x)$$

where $f(D) = D^2 - 4D + 3$

$Q(x) = 4e^{3x}$

A.E is $f(m)=0 \Rightarrow m^2-4m+3=0$

$\Rightarrow m=3\ ;\ m=1.$

$C.F = y_c = c_1e^{3x}+c_2e^{x}$ — (1)

$P.I = y_p = \frac{1}{f(D)}Q(x)$

$= \frac{1}{D^2-4m+3}4e^{3x}$

$= \frac{1}{(D-3)(D-1)}4e^{3x}$

$= \frac{4e^{3x}}{2}\cdot\frac{x}{1!}$

$= 2xe^{3x}$

$\therefore\ y = y_c+y_p$

$y = c_1e^{3x}+c_2e^{x}+2xe^{3x}$ — (2)

Given that $y(0)=-1$ put in (2)

$-1 = c_1+c_2+0 \Rightarrow \boxed{c_1+c_2=-1}$ — (3)

Given that $y'(0)=3$ put in (4)

Diff. (2) w.r.to 'x' on b.s.

$y' = 3c_1e^{3x}+c_2e^{x}+6xe^{3x}+2e^{3x}$ — (4)

$3 = 3c_1+c_2+2 \Rightarrow \boxed{3c_1+c_2=1}$ — (5)

solving (3) & (5)

$c_1+\cancel{c_2}=-1$
$3c_1+\cancel{c_2}=1$

$-2c_1 = -2 \Rightarrow \boxed{c_1=1}$ & $\boxed{c_2=-2}$

$\therefore$ from (2)

$y = 1e^{3x}+(-2)e^{x}+2xe^{3x}$

$y = e^{3x}-2e^{x}+2xe^{3x}$

(11). Solve $y''+4y'+4y = 4\cos x + 3\sin x$, $y(0)=0$, $y'(0)=0$.

Sol: $(D^2+4D+4)y = 4\cos x + 3\sin x$.

A.E is $f(m)=0 \Rightarrow m^2+4m+4=0 \Rightarrow (m+2)^2=0$
$\Rightarrow m=-2,-2$

$\therefore y_c = (c_1+c_2x)e^{mx}$
$= (c_1+c_2x)e^{-2x}$ —— (1)

P.I is $\frac{1}{f(D)}\phi(x)$

$= \frac{1}{D^2+4D+4}(4\cos x + 3\sin x)$

$= \frac{4\cos x}{D^2+4D+4} + \frac{3\sin x}{D^2+4D+4}$

$= \frac{4\cos x}{4D+3} + \frac{3\sin x}{4D+3}$

$= \frac{(4\cos x + 3\sin x)}{4D+3} \times \frac{4D-3}{4D-3} = \frac{-16\sin x + 12\cos x + 12\cos x - 9\sin x}{-25}$

$= \sin x$ //

$\boxed{\therefore y = (c_1+c_2x)e^{-2x} + \sin x \text{ —— (2)}}$

Given that $y(0)=0 \Rightarrow 0 = c_1 + c_2(0) \Rightarrow \boxed{c_1=0}$

and $\boxed{y' = (c_1+c_2x)(-2)e^{-2x} + e^{-2x}(c_2) + \cos x}$

Given that $y'(0)=0 \Rightarrow 0 = c_1(-2) + c_2 + 1 \Rightarrow \boxed{c_2=-1}$.

$\therefore y = -xe^{-2x} + \sin x$ //

HW

(12). Solve the D.E $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x$, Given $y(0)=0$, $y'(0)=1$.

(A:- $\left(y = e^{-2x}\left(3/5\cos x + 9/5\sin x\right) - \frac{e^{x}}{10} - \frac{e^{-x}}{2}\right)$ //

case(2) :- P.I of $f(D)y = \phi(x)$, when $\phi(x) = \sin ax$ (or) $\phi(x) = \cos ax$, where 'a' is any constant

If $\phi(x) = \sin ax$ (or) $\cos ax$ then

$$P.I = \frac{1}{f(D)}\phi(x)$$

$$= \frac{1}{f(D)}\sin ax \text{ (or) } \frac{1}{f(D)}\cos ax.$$

step(1) :- Replace 'D^2' by '$-a^2$'

'D^3' by $D^2 \cdot D = -a^2 D$

'D^4' by $D^2 \cdot D^2 = a^4$

- - - - - - - - -

and soon.

step(2) :- Rationalise the denominator, then sub. put $D^2 = -a^2$ ---- etc,

step(3) :- Differentiate and simplify.

* Problems *

① Find P.I of $(D^2+1)y = e^{2x} + \sin 3x$.

Sol:- $P.I = \frac{1}{(D^2+1)}\left(e^{2x} + \sin 3x\right) = \frac{e^{2x}}{D^2+1} + \frac{\sin 3x}{D^2+1}$

$$= \frac{e^{2x}}{5} + \frac{\sin 3x}{-9+1}$$

($\because$ put $D = a$, $D = 2$ & put $D^2 = -a^2$, $D^2 = -3^2$, $D^2 = -9$)

$$= \frac{e^{2x}}{5} - \frac{\sin 3x}{8}$$ //

(2) Solve $(D^2+3D+2)y = \sin 3x$.

Sol: $y_c = c_1 e^{-2x} + c_2 e^{-x}$

$$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^2+3D+2} \sin 3x$$

$$= \frac{1}{-9+3D+2} \sin 3x \quad \begin{pmatrix} \text{put} \\ \because D^2 = -a^2 \\ D^2 = -9 \end{pmatrix}$$

$$= \frac{1}{3D-7} \sin 3x$$

$$= \left(\frac{1}{3D-7} \times \frac{3D+7}{3D+7}\right) \sin 3x$$

$$= \left(\frac{3D+7}{9D^2-49}\right) \sin 3x$$

$$= \left(\frac{3D+7}{9(-9)-49}\right) \sin 3x$$

$$= \left(\frac{3D+7}{-130}\right) \sin 3x$$

$$= -\frac{1}{130}\left[3D \sin 3x + 7 \sin 3x\right]$$

$$= -\frac{1}{130}\left[9\cos 3x + 7 \sin 3x\right]$$

$\therefore$ $y = y_c + y_p$

$$= c_1 e^{-2x} + c_2 e^{-x} - \frac{1}{130}\left(9\cos 3x + 7\sin 3x\right)$$

where c_1, c_2 are constants.

(24)

(3) Solve $(D^2-3D+2)y = \cos 3x$.

Sol: $y_c = c_1e^{x}+c_2e^{2x}$.

$y_p = \frac{1}{D^2-3D+2}(\cos 3x)$

$= \frac{1}{-3D-7}\cos 3x \quad \left(\because D^2=-a^2,\ D^2=-9\right)$

$= \frac{-1}{3D+7}\cos 3x = \left(\frac{-1}{3D+7} \times \frac{3D-7}{3D-7}\right)\cos 3x$

$= \frac{-(3D-7)}{9D^2-49}\cos 3x = \frac{-(3D-7)}{-81-49}\cos 3x = \frac{3D-7}{130}(\cos 3x)$

$= \frac{1}{130}\left[3(-3\sin 3x)-7\cos 3x\right] = \frac{1}{130}(-9\sin 3x - 7\cos 3x)$

$= -\frac{1}{130}(9\sin 3x + 7\cos 3x)$.

$\therefore \boxed{y = y_c + y_p}$

$y = c_1e^{x}+c_2e^{2x} - \frac{1}{130}(9\sin 3x + 7\cos 3x)$

(4). $(D^2-4)y = 2\cos^2 x$.

Sol:- $(D^2-4)y = \cancel{2}\frac{(1+\cos 2x)}{\cancel{2}} \quad (\because \cos 2\theta = 2\cos^2\theta - 1)$

$(D^2-4)y = 1+\cos 2x$

$f(D)y = Q(x)$.

$\therefore y_c = c_1e^{2x}+c_2e^{-2x}$.

$y_p = \frac{1}{D^2-4}(1+\cos 2x) = \frac{1}{D^2-4}+\frac{\cos 2x}{D^2-4}. = \frac{e^{0.x}}{D^2-4}+\frac{\cos 2x}{D^2-4}.$

(put $D=0$) (put $D^2=-a^2$, $D^2=-2^2$, $D^2=-4$)

$= \frac{1}{-4}+\frac{\cos 2x}{-4-4} = \frac{-1}{4}-\frac{\cos 2x}{8}.$

$\boxed{\therefore y = c_1e^{2x}+c_2e^{-2x} - 1/4 - \frac{\cos 2x}{8}}$

⑤. Solve $(D^2-4D+3)y = \sin 3x \cos 2x$.

Sol: $y_c = c_1e^{x}+c_2e^{3x}$ —— (1)

$$y_p = \frac{1}{D^2-4D+3}(\sin 3x \cos 2x)$$

$$y_p = \frac{1}{2}\frac{1}{D^2-4D+3}(2\sin 3x \cdot \cos 2x)$$

$$= \frac{1}{2}\frac{1}{D^2-4D+3}(2\sin 3x \cos 2x)$$

$\left[\because 2\sin A\cos B = \sin(A+B)+\sin(A-B)\right.$
$2\cos A\sin B = \sin(A+B)-\sin(A-B)$
$2\cos A\cos B = \cos(A+B)+\cos(A-B)$
$-2\sin A\sin B = \cos(A+B)-\cos(A-B)$
(or)
$\left.2\sin A\sin B = \cos(A-B)-\cos(A+B)\right]$

$$= \frac{1}{2}\frac{1}{D^2-4D+3}(\sin 5x+\sin x)$$

$$= \frac{1}{2}\frac{\sin 5x}{D^2-4D+3}+\frac{1}{2}\frac{\sin x}{D^2-4D+3}$$

$$= \frac{1}{2}\frac{\sin 5x}{-25-4D+3}+\frac{1}{2}\frac{\sin x}{-1^2-4D+3}$$

$$= \frac{1}{2}\frac{\sin 5x}{-4D-22}+\frac{1}{2}\frac{\sin x}{-4D+2}$$

$$= -\frac{1}{4}\frac{\sin 5x}{2D+11}\times\frac{2D-11}{2D-11}+\frac{1}{4}\frac{\sin x}{1-2D}\times\frac{1+2D}{1+2D}$$

$$= -\frac{1}{4}\frac{(2D-11)\sin 5x}{4D^2-121}+\frac{1}{4}\frac{(1+2D)\sin x}{1-4D^2}$$

$$= -\frac{1}{4}\left(\frac{2\cos 5x(5)-11\sin 5x}{-100-121}\right)+\frac{1}{4}\frac{\sin x+2\cos x}{5}$$

$$= \frac{1}{4}\left(\frac{10\cos 5x-11\sin 5x}{221}\right)+\frac{1}{20}(\sin x+2\cos x)$$

$$= \frac{1}{884}(10\cos 5x-11\sin 5x)+\frac{1}{20}(\sin x+2\cos x) \quad \text{—— (2)}$$

$\therefore \boxed{y = y_c+y_p}$.

$\boxed{\text{Note}}$:– ①. $\frac{\sin ax}{D^2+a^2} = \frac{-x}{2a}\cos ax$

②. $\frac{\cos ax}{D^2+a^2} = \frac{x}{2a}\sin ax$.

⑤ Solve $(D^2+1)y = \sin x \sin 2x$

Sol? $y_c = c_1\cos x + c_2 \sin x$

$y_p = \frac{1}{D^2+1}(\sin x \sin 2x)$

$= \frac{1}{2(D^2+1)}(2\sin x \sin 2x)$

$= \frac{1}{2(D^2+1)}(\cos x - \cos 3x)$

$= \frac{\cos x}{2(D^2+1)} - \frac{\cos 3x}{2(D^2+1)}$

$= \frac{1}{2}\frac{x}{2(1)}\sin x - \frac{1}{2}\frac{1}{(8)}\cos 3x \left(\because \frac{\cos ax}{D^2+a^2} = \frac{x}{2a}\sin ax\right)$

$y_p = \frac{x}{4}\sin x + \frac{\cos 3x}{16}$.

$\therefore y = y_c + y_p$

$y = c_1\cos x + c_2 \sin x + \frac{x\sin x}{4} + \frac{\cos 3x}{16}$ //.

H.W

⑥. Solve $(D^2+4)y = \cos 2x$.

Sol? $y = c_1\cos 2x + c_2\sin 2x + \frac{x\sin 2x}{4}$ //

HW

⑦. Solve $y'' + 4y' + 20y = 23\sin t - 15\cos t$, $y(0) = 0$, $y'(0) = -1$ (27)

A:- $\begin{pmatrix} c_1 = 1 \\ c_2 = 0 \end{pmatrix}$ & $\left(y = e^{-2t}\cos 4t + \sin t - \cos t\right)$

⑧. Solve $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$.

Sol:- $(D^3 + 4D)y = \sin 2x$ —— (1)

$y_c = c_1 + c_2\cos 2x + c_3\sin 2x$ —— (2)

$y_p = \frac{1}{D(D^2+4)}\sin 2x$

$= \frac{-\cos 2x}{2(D^2+4)}$

$= -\frac{1}{2}\left[\frac{x}{4}\sin 2x\right]$

$y_p = \frac{-x\sin 2x}{8}$ —— (3)

$\therefore y = y_c + y_p$

$= c_1 + c_2\cos 2x + c_3\sin 2x + \frac{x\sin 2x}{8}$ //

HW

⑨ Solve $(D^3 + 4D)y = 5 + \sin 2x$

Ans:- $y_c = c_1 + c_2\cos 2x + c_3\sin 2x$

$y_p = \frac{1}{D(D^2+4)}(5 + \sin 2x)$

$= \frac{5 \cdot e^{0.x}}{D(D^2+4)} + \frac{\sin 2x}{D(D^2+4)}$

$= \frac{5}{D(4)} + \frac{-\cos 2x}{2(D^2+4)}$

$= \frac{5}{4}x - \frac{1}{8}x\sin 2x$ //

$\therefore \boxed{y = y_c + y_p}$

(10). solve $(D^2+9)y = \cos 3x + \sin 2x$ (28)

Sol: $y_c = c_1\cos 3x + c_2\sin 3x$

$$y_p = \frac{1}{D^2+9}(\cos 3x + \sin 2x)$$

$$= \frac{\cos 3x}{D^2+9} + \frac{\sin 2x}{D^2+9}$$

$$= \frac{x}{6}\sin 3x + \frac{\sin 2x}{5} \quad (\because \text{put } D^2 = -4)$$

$$\therefore \boxed{y = y_c + y_p}$$

(11) solve $(D^3+1)y = \cos(2x-1)$

Sol: $y_c = c_1 e^{-x} + e^{x/2}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$

$$y_p = \frac{1}{D^3+1}\cos(2x-1)$$

$$= \frac{1}{D^2 \cdot D+1}\cos(2x-1) \quad (D^2 = -4)$$

$$= \frac{1}{-4D+1}\cos(2x-1)$$

$$= \left(\frac{1}{1-4D} \times \frac{1+4D}{1+4D}\right)\cos(2x-1)$$

$$= \frac{1+4D}{1-16D^2}\cos(2x-1)$$

$$= \frac{1+4D}{65}\cos(2x-1)$$

$$y_p = \frac{1}{65}\left[\cos(2x-1) - 8\sin(2x-1)\right]$$

$$\boxed{y = y_c + y_p}$$

(b). Solve. $(D^4 - 2D^3 + 2D^2 - 2D + 1)\,y = \cos x$.

Ans:

$$y_c = (c_1 + c_2 x)e^{x} + (c_3 \cos x + c_4 \sin x)$$

$$y_p = \frac{1}{D^4 - 2D^3 + 2D^2 - 2D + 1}\cos x$$

$m=1$	1	-2	2	-2	1
	0	1	-1	1	-1
$m=1$	1	-1	1	-1	$\boxed{0}$
	0	1	0	1	
	1	0	1	$\boxed{0}$	

$$= \frac{1}{(D^4 - 2D^3) + D^2 + D^2 - 2D + 1}\cos x$$

$$= \frac{1}{D^2(D^2 - 2D + 1) + (D^2 - 2D + 1)}\cos x$$

$$= \frac{1}{(D^2 - 2D + 1)(D^2 + 1)}\cos x$$

$$= \frac{1}{-2D(D^2 + 1)}\cos x \qquad (D^2 = -1)$$

$$= -\frac{1}{2}\,\frac{\sin x}{D^2 + 1}$$

$$= +\frac{1}{2}\,\frac{x}{2}\cos x$$

$$y_p = +\frac{x}{4}\cos x.$$

$\therefore\ y = y_c + y_p$

$$y = (c_1 + c_2 x)e^{x} + (c_3 \cos x + c_4 \sin x) + \frac{x}{4}\cos x$$

Case (3). – P.I of $f(D)y = \phi(x)$, where $\phi(x) = x^k$, where 'k' is a +ve Integer.

$$\therefore\ P.I = \frac{1}{f(D)}\phi(x)$$

$$= \frac{1}{f(D)}x^k$$

Working rule to find P.I :—

Step (1) :- From $f(D)$ take out the lowest degree term then common that term then $f(D)$ convert into $[1 \pm g(D)]^n$ form.

Step (2) :- we take $[1 \pm g(D)]^n$ in the numerator, so that it takes the form $[1 \pm g(D)]^{-n}$.

Step (3) :- Expand $[1 + g(D)]^{-n}$ by Binomial thm.

Step (4) :- Differentiate term by term.

Note :- By Binomial Expansions

(i). $(1+D)^{-1} = 1 - D + D^2 - D^3 + D^4 - D^5 + \cdots$

(ii) $(1-D)^{-1} = 1 + D + D^2 + D^3 + D^4 + D^5 + \cdots$

(iii) $(1+D)^{-2} = 1 - 2D + 3D^2 - 4D^3 + 5D^4 \cdots$

(iv) $(1-D)^{-2} = 1 + 2D + 3D^2 + 4D^3 + 5D^4 + \cdots$

(v). $(1-D)^{-3} = 1 + 3D + 6D^2 + 10D^3 + 15D^4 + \cdots$

(vi) $(1+D)^{-3} = 1 - 3D + 6D^2 - 10D^3 + 15D^4 - \cdots$

Problems :

① P.I of $(D^2 + 3D + 2)y = x$.

Sol: $P.I = \dfrac{1}{f(D)}\phi(x) = \dfrac{1}{D^2 + 3D + 2}x^1$.

$$= \frac{1}{2\left[1+\frac{D^2+3D}{2}\right]}x$$

$$= \frac{x}{2}\left[1+\frac{D^2+3D}{2}\right]^{-1}$$

$$= \frac{x}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2-\cdots\cdots\right]$$

$$= \frac{x}{2}\left[1-\frac{3D}{2}\right] \quad (\text{neglecting } D^2, D^3, \text{--- etc, terms})$$

$$= \frac{x}{2}-\frac{3}{4}(1)$$

$$y_p = \frac{x}{2}-3/4$$

(27). solve $(D^2+D+1)y = x^3$.

Sol: $y_c = e^{-x/2}\left(c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right)$

$$y_p = \frac{1}{(D^2+D+1)}x^3$$

$$= \frac{1}{[1+(D^2+D)]}x^3 = \frac{1}{[1+(D^2+D)]^1}x^3$$

$$= [1+(D^2+D)]^{-1}x^3$$

$$= \left[1-(D^2+D)+(D^2+D)^2-(D^2+D)^3\right]x^3 \quad (\text{neglecting } D^4, D^5 \text{ --- terms})$$

$$= \left[1-(\cancel{D^2}+D)+(D^4+\cancel{D^2}+2D^3)-D^3-\cdots\right]x^3$$

$$= \left[x^3-D(x^3)+D^3(x^3)\right]$$

$$\boxed{y_p = x^3-3x^2+6}$$

$$\boxed{\therefore y = y_c+y_p}$$

(32).

3. Solve $(D^3+2D^2+D)y = x^3$.

Sol: $y_c = c_1 + (c_2+c_3x)\cdot e^{-x}$.

$$y_p = \frac{1}{D(D^2+2D+1)} x^3$$

$$= \frac{1}{D(D+1)^2} x^3$$

$$= \frac{1}{(D+1)^2} \frac{x^4}{4}$$

$$= \frac{1}{4}(D+1)^{-2} x^4$$

$$= \frac{1}{4}\left[1-2D+3D^2-4D^3+5D^4 - - -\right] x^4 \quad (\because \text{neglecting } D^5, D^6 \text{ ---- terms})$$

$$= \frac{1}{4}\left(x^4 - 8x^3 + 36x^2 - 96x + 120\right)$$

$$= \frac{x^4}{4} - \frac{8x^3}{4} + \frac{36x^2}{4} - \frac{96x}{4} - \frac{120}{4}$$

$$y_p = \frac{x^4}{4} - 2x^3 + 9x^2 - 24x + 30$$

$$\therefore \boxed{y = y_c + y_p}$$

4. Solve $(D^3-3D-2)y = x^2$

Sol: $y_c = c_1e^{2x} + (c_2+c_3x)e^{-x}$

$$y_p = \frac{1}{D^3-3D-2} x^2$$

$$= \frac{1}{-2\left[1-\left(\frac{D^3-3D}{2}\right)\right]} x^2 = \frac{-x^2}{2}\left[1-\left(\frac{D^3-3D}{2}\right)\right]^{-1}$$

$$= -\frac{1}{2}\left[1-\left(\frac{D^3-3D}{2}\right)\right]^{-1} x^2$$

$$= -\frac{1}{2}\left[1+\left(\frac{D^3-3D}{2}\right)+\left(\frac{D^3-3D}{2}\right)^2+ \cdots\right] x^2$$

$$= -\frac{1}{2}\left[1+\frac{D^3-3D}{2}+\frac{D^6+9D^2-6D^4}{4}+\cdots\right] x^2$$

$$= -\frac{1}{2}\left[x^2-\frac{3D}{2}x^2+\frac{9}{4}D^2x^2 \cdots\right]$$

$$= -\frac{1}{2}\left[x^2-\frac{3}{2}(2x)+\frac{9}{4}(2)\right]$$

$$y_p = -\frac{1}{2}\left(x^2-3x+\frac{9}{2}\right).$$

$$\boxed{\therefore y = y_c+y_p}.$$

H.W.

(5). Solve $\frac{d^2y}{dx^2}+\frac{dy}{dx} = x^2+2x+4.$

Sol? $\left[y = y_c+y_p = c_1+c_2e^{-x}+\frac{x^3}{3}+4x\right].$

(6). Solve $D^2(D^2+4)y = 96x^2+\sin 2x - k.$

Sol? $y_c = (c_1+c_2x)+c_3\cos 2x+c_4\sin 2x$

$$y_p = \frac{1}{D^2(D^2+4)}\left(96x^2+\sin 2x - k\right)$$

$$= \frac{1}{4D^2\left(1+\frac{D^2}{4}\right)}\left(96x^2+\sin 2x - k\right)$$

$$= \frac{1}{4D^2\left(1+\frac{D^2}{4}\right)}96x^2 + \frac{1}{D^2(D^2+4)}\sin 2x - \frac{1}{D^2(D^2+4)}(k)$$

$$= \frac{1}{4D^2}\left(1+\frac{D^2}{4}\right)^{-1}96x^2 + \frac{1}{(-4)(D^2+2^2)}\sin 2x - \frac{k e^{0 \cdot x}}{D^2(D^2+4)}$$

$$= \frac{1}{4D^2}\left(1-\frac{D^2}{4}+\frac{D^4}{16}-\cdots\right)96x^2+\frac{1}{4}\frac{x}{2(2)}\cos 2x-\frac{k.e^{0.x}}{D^2(4)}$$

$$= \frac{1}{4D^2}\left[96x^2-\frac{1}{4}(192)\right]+\frac{x}{16}\cos 2x-\frac{k}{4}\frac{x^2}{2!}$$

$$= \frac{1}{D^2}\left(24x^2-12\right)+\frac{x}{16}\cos 2x + \frac{kx^2}{8}$$

$$= 24\frac{x^4}{12}-12\frac{x^2}{2}+\frac{x}{16}\cos 2x+\frac{kx^2}{8}$$

$$= \left(2x^4-6x^2\right) + \frac{x}{16}\cos 2x+\frac{k}{8}x^2$$

H.W

(7). $(D^3-3D-2)y = x^3$.

$$\left[y = y_c+y_p = (c_1+c_2x)e^{-x}+c_3e^{2x}-\frac{1}{8}\left(4x^3+18x^2+54x+93\right)\right]$$

(8). Solve $(D^3-3D^2+3D-1)y = \sin x + x^3$.

Sol $y_c = (c_1+c_2x+c_3x^2)e^x$.

$$y_p = \frac{1}{D^3-3D^2+3D-1}\sin x + \frac{x^3}{D^3-3D^2+3D-1}$$

$$= \frac{1}{-D+3+3D-1}\sin x + \frac{x^3}{(D-1)^3}$$

$$= \frac{1}{2D+2}\sin x + (D-1)^{-3}x^3$$

$$= \frac{1}{2(D+1)}\frac{(D-1)}{(D-1)}\sin x - (1-D)^{-3}x^3$$

$$= \frac{D-1}{2(D^2-1)}\sin x - \left[1+3D+6D^2+10D^3+\cdots\right]x^3$$

$$= \frac{\cos x-\sin x}{-4}-x^3-9x^2-36x-60 = \frac{\sin x-\cos x}{4}-x^3-9x^2-36x-60$$

9. solve $(D^2+3D+2)y = 2\cos(2x+3) + 2e^x + x^2$

Sol: $y_c = c_1 e^{-x} + c_2 e^{-2x}$

$$y_p = \frac{1}{D^2+3D+2} 2\cos(2x+3) + \frac{2e^x}{D^2+3D+2} + \frac{x^2}{D^2+3D+2}$$

$$= 2\frac{1}{3D-2}\cos(2x+3) + \frac{1}{3}e^x + \left[1+\left(\frac{D^2+3D}{2}\right)\right]^{-1} x^2$$

$$= \frac{2(3D+2)}{9D^2-4}\cos(2x+3) + \frac{1}{3}e^x + \left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2 - \cdots\right]x^2$$

$$= \frac{-1}{20}\left(-6\sin(2x+3)+2\cos(2x+3)\right) + \frac{1}{3}e^x + \frac{1}{2}\left[x^2 - \frac{1}{2}(2+6x) + \frac{9}{4}(2)\right]$$

$$= \frac{-1}{10}\left[\cos(2x+3) - 3\sin(2x+3)\right] + \frac{1}{3}e^x + \frac{1}{2}\left[x^2 - 3x + \frac{7}{2}\right]$$

$$\therefore \boxed{y = y_c + y_p}$$

10. solve $y''' + 2y'' - y' - 2y = 1 - 4x^3$.

Sol: $y_c = c_1 e^x + c_2 e^{-x} + c_3 e^{-2x}$.

$$y_p = \frac{1}{D^3+2D^2-D-2}(1-4x^3)$$

$$= \frac{1}{-2\left[1-\left(\frac{D^3+2D^2-D}{2}\right)\right]}(1-4x^3)$$

$$= -\frac{1}{2}\left[1-\left(\frac{D^3+2D^2-D}{2}\right)\right]^{-1}(1-4x^3)$$

$$= -\frac{1}{2}\left[1+\left(\frac{D^3+2D^2-D}{2}\right)+\left(\frac{D^3+2D^2-D}{2}\right)^2+\left(\frac{D^3+2D^2-D}{2}\right)^3+\cdots\right](1-4x^3)$$

$$= -\frac{1}{2}\left[1+\frac{1}{2}\left(D^3+2D^2-D\right)+\frac{1}{4}\left(D^2-4D^3\right)+\frac{1}{8}\left(-D^3\right)\right]\left(1-4x^3\right)$$

$$= -\frac{1}{2}\left[1+D^3\left(\frac{1}{2}-1-\frac{1}{8}\right)+D^2\left(1+\frac{1}{4}\right)-\frac{1}{2}D\right]\left(1-4x^3\right)$$

$$= -\frac{1}{2}\left[1-\frac{5}{8}D^3+\frac{5}{4}D^2-\frac{1}{2}D\right]\left(1-4x^3\right)$$

$$= -\frac{1}{2}\left[\left(1-4x^3\right)-\frac{5}{8}(-24)+\frac{5}{4}\left(-24x\right)-\frac{1}{2}\left(-12x^2\right)\right]$$

$$= -\frac{1}{2}\left[-4x^3+6x^2-30x+16\right]$$

$$y_p = 2x^3-3x^2+15x-8$$

$$\therefore \boxed{y = y_c + y_p}$$

HW

(11). $(D^2-4D+4)y = 8x^2+e^{2x}$

Ans. $y = y_c+y_p = e^{2x}(c_1+c_2x)+\frac{x^2}{2}e^{2x}+2x^2+4x+3$

HW

(12). $(D^2-4)y = x\sinh x+54x+8$

Ans. $y_c = c_1e^{2x}+c_2e^{-2x}$

$$y_p = \frac{1}{D^2-4}\left(x\sinh x+54x+8\right)$$

$$= \frac{xe^{x}}{2(D^2-4)}-\frac{xe^{-x}}{2(D^2-4)}+\frac{54x}{D^2-4}+\frac{8}{D^2-4}$$

$\left(\because \sinh x = \frac{e^{x}-e^{-x}}{2}\right)$

$$= \left(y_{p_1}+y_{p_2}+y_{p_3}+y_{p_4}\right)$$

(∵ case(4) $= y_{p_1}$

case(4) $= y_{p_2}$

case(3) $= y_{p_3}$

case(1) $= y_{p_4}$)

Now $y_{p_1} = \frac{xe^x}{2(D^2-4)} = \frac{e^x}{2}\left[\frac{x}{(D+1)^2-4}\right]$

$= \frac{e^x}{2}\left(\frac{x}{D^2+2D-3}\right) = \frac{e^x}{-6}\frac{x}{\left[1-\left(\frac{D^2+2D}{3}\right)\right]}$

$\Rightarrow y_{p_1} = \frac{e^x}{-6}\left[1-\left(\frac{D^2+2D}{3}\right)\right]^{-1}\cdot x = \frac{-e^x}{6}\left(1+\frac{D^2+2D}{3}\right)x = -\frac{e^x}{6}\left(x+\frac{2}{3}\right)$ —— (1)

Now $y_{p_2} = \frac{-xe^{-x}}{2(D^2-4)} = -\frac{e^{-x}}{2}\left[\frac{x}{(D-1)^2-4}\right]$

$= -\frac{e^{-x}}{2}\left[\frac{x}{D^2-2D-3}\right] = \frac{e^{-x}}{6}\frac{x}{\left[1-\left(\frac{D^2-2D}{3}\right)\right]}$

$\Rightarrow y_{p_2} = \frac{e^{-x}}{6}\left[1-\left(\frac{D^2-2D}{3}\right)\right]^{-1}x = \frac{e^{-x}}{6}\left(1+\frac{D^2-2D}{3}\right)x = \frac{e^{-x}}{6}\left(x-\frac{2}{3}\right)$ —— (2)

Now $y_{p_3} = \frac{54x}{D^2-4} = \frac{54x}{-4\left(1-\frac{D^2}{4}\right)} = -\frac{27}{2}\left(1-\frac{D^2}{4}\right)^{-1}x$

$\rightarrow y_{p_3} = -\frac{27}{2}\left(1+\frac{D^2}{4}\right)x = -\frac{27}{2}x$ —— (3)

Now $\rightarrow y_{p_4} = \frac{8}{D^2-4} = \frac{8\cdot e^{0\cdot x}}{D^2-4} = \frac{8}{0-4} = -2$ —— (4)

$\therefore y_p = y_{p_1} + y_{p_2} + y_{p_3} + y_{p_4}$

$y_p = -\frac{e^x}{6}\left(x+\frac{2}{3}\right) + \frac{e^{-x}}{6}\left(x-\frac{2}{3}\right) - 2 - \frac{27}{2}x$ //

$\therefore \boxed{y = y_c + y_p}$

(13). P.I. of $(D+1)^2y = e^{-x}+x^2$ $\left[A:- \frac{e^{-x}\cdot x^2}{2} + (x^2-4x+6)\right]$ //

Case(4) :- P.I of $f(D)y = \phi(x)$ when $\phi(x) = e^{ax} \cdot V$, where (38).

a = constant

V = function of 'x'.

where $V = \sin ax$ (or) $\cos ax$ (or) x^k.

$$\therefore P.I = \frac{1}{f(D)} \phi(x)$$

$$= \frac{1}{f(D)} e^{ax} \cdot V$$

$$= e^{ax} \cdot \left[\frac{1}{f(D+a)}\right] V.$$

& $\frac{1}{f(D+a)} \cdot V$ is Evaluated depending on 'V'.

Problems :- (1). Solve $(D^2+2)y = e^x \cos x$.

Sol:

$$y_c = c_1 e^{x} + c_2 e^{-x} + c_3 \cos\sqrt{2}x + c_4 \sin\sqrt{2}x.$$

$$y_p = \frac{1}{(D^2+2)} e^x \cos x$$

$$= e^x \frac{1}{[(D+1)^2+2]} \cos x \qquad \left[\because \frac{1}{f(D)} e^{ax} \cdot V = e^{ax} \frac{1}{f(D+a)} V\right]$$

$$= e^x \left(\frac{1}{D^2+2D+3}\right) \cos x = e^x \left(\frac{1}{-1^2+2D+3}\right) \cos x \quad (\because \text{put } D^2 = -1)$$

$$= e^x \left(\frac{1}{2D+2}\right) \cos x = \frac{e^x}{2} \left(\frac{D-1}{D^2-1}\right) \cos x = \frac{+e^x}{-4} (\sin x - \cos x)$$

$$= \frac{+e^x}{4} (\sin x + \cos x)$$

$$\therefore \boxed{y = y_c + y_p}$$

(39)

Q. Solve $(D^2-2D+1)y = x^2e^{3x} - \sin 2x + 3$.

Sol: $y_c = (c_1+c_2x)e^{x}$.

$$y_p = \frac{1}{D^2-2D+1}\left(x^2e^{3x} - \sin 2x + 3\right)$$

$$= e^{3x}\left[\frac{1}{(D+3)^2-2(D+3)+1}\right]x^2 - \frac{\sin 2x}{D^2-2D+1} + \frac{3}{D^2-2D+1}$$

$$= e^{3x}\left(\frac{1}{D^2+9+6D-2D-6+1}\right)x^2 - \frac{\sin 2x}{-2D-3} + \frac{3e^{0.x}}{D^2-2D+1}$$

$$= e^{3x}\left(\frac{1}{D^2+4D+4}\right)x^2 + \frac{\sin 2x}{(2D+3)} \times \frac{2D-3}{2D-3} + \frac{3}{1}$$

$$= \frac{e^{3x}}{4}\frac{1}{\left[1+\left(\frac{D^2+4D}{4}\right)\right]}x^2 + \frac{(2D-3)\sin 2x}{4D^2-9} + 3$$

$$= \frac{e^{3x}}{4}\left[1+\left(\frac{D^2+4D}{4}\right)\right]^{-1}x^2 + \frac{4\cos 2x - 3\sin 2x}{(-25)} + 3$$

$$= \frac{e^{3x}}{4}\left[1-\left(\frac{D^2+4D}{4}\right)+\left(\frac{D^2+4D}{4}\right)^2\right]x^2 - \frac{1}{25}\left(4\cos 2x - 3\sin 2x\right) + 3$$

$$= \frac{e^{3x}}{4}\left[1-\frac{D^2}{4}+D+\frac{D^4}{4}+\frac{16D^2}{16}+\frac{2D^3(4)}{16}\right]x^2 - \text{''} + 3$$

$$= \frac{e^{3x}}{4}\left(x^2-\frac{1}{4}(2)-2x+0+2+0\right) - \text{''} + 3$$

$$= \frac{e^{3x}}{4}\left(x^2-\frac{1}{2}-2x+2\right) - \text{''} + 3$$

$$= \frac{e^{3x}}{4}\left(x^2-2x+\frac{3}{2}\right) - \text{''} + 3$$

$$y_p = \frac{e^{3x}}{8}\left(2x^2-4x+3\right) - \text{''} + 3.$$

$\therefore \boxed{y = y_c + y_p}$

3. $(D^2-2D+4)y = e^{2x}\cos x$.

Sol:

$y_c = e^{x}(c_1 \sin\sqrt{3}x + c_2 \cos\sqrt{3}x)$

$y_p = \dfrac{1}{D^2-2D+4} e^{2x}\cos x$

$= e^{2x} \dfrac{1}{(D+2)^2-2(D+2)+4} \cos x$

$= e^{2x} \dfrac{1}{D^2+4+4D-2D-4+4} \cos x$

$= e^{2x} \dfrac{1}{D^2+2D+4} \cos x$

$= e^{2x} \dfrac{1}{2D+3} \cos x = e^{2x} \dfrac{(2D-3)\cos x}{4D^2-9} = \dfrac{e^{2x}(-2\sin x - 3\cos x)}{-13}$

$y_p = \dfrac{e^{2x}}{13}(2\sin x + 3\cos x)$

$\therefore \boxed{y = y_c + y_p}$

H.W

4. $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 6y = e^{2x}(1+x)$. $\left[y = c_1e^{x} + c_2e^{6x} - \dfrac{e^{2x}}{16}(4x+1)\right]$

5. $(D^3-7D^2+14D-8)y = e^{x}\cos 2x$

Sol:

$y_c = c_1e^{x} + c_2e^{2x} + c_3e^{4x}$

$y_p = \dfrac{1}{D^3-7D^2+14D-8} e^{x}\cos 2x$

$= e^{x}\dfrac{1}{(D+1)^3 - 7(D+1)^2 + 14(D+1) - 8} \cos 2x$

$= e^{x} \dfrac{1}{D^3-4D^2+3D} \cos 2x$

$= e^{x} \dfrac{1}{-4D+16+3D} \cos 2x$

$$= e^{x}\frac{1}{16-D}\cos 2x = e^{x}\frac{16+D}{260}\cos 2x$$

$$y_p = \frac{e^{x}}{260}\left(16\cos 2x - 2\sin 2x\right)$$

$$\therefore \boxed{y = y_c + y_p}$$

H.W

6. $(D^3-4D^2-D+4)y = e^{3x}\cos 2x$ $\left[\because c_1e^{x}+c_2e^{-x}+c_3e^{4x}-\frac{e^{3x}}{20}(\sin 2x + 7\cos 2x)\right]$

7. Solve $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 13y = 8e^{3x}\sin 2x.$

Sol:

$$y_c = e^{3x}(c_1\cos 2x + c_2\sin 2x)$$

$$y_p = \frac{1}{D^2-6D+13}\left(8e^{3x}\sin 2x\right)$$

$$= 8e^{3x}\frac{1}{(D+3)^2-6(D+3)+13}\sin 2x$$

$$= 8e^{3x}\frac{1}{D^2+4}\sin 2x$$

$$= 8e^{3x}\frac{-x\cos 2x}{4}$$

$$y_p = -2x\cos 2x\, e^{3x}$$

$$\therefore \boxed{y = y_c + y_p}$$

H.W

8. Solve $(D^2+1)y = \sin x\sin 2x + e^{x}x^2$

$\left[y = c_1\cos x + c_2\sin x + \frac{x}{4}\sin x + \frac{1}{16}\cos 3x + \frac{e^{x}}{2}(x^2-2x+1)\right]$

Case(5):- If $\phi(x) = x^m \cdot V$, where 'm' is a +ve Integer and 'V' is any function of 'x' then P.I is

$$P.I = \frac{1}{f(D)} \phi(x)$$

$$= \frac{1}{f(D)} x^m \cdot V$$

But Here 'V' is of the form $\sin ax$ (or) $\cos ax$ can be Evaluated as follows.

Put $\phi(x) = x^m \sin ax$:-

$$P.I = \frac{1}{f(D)} \phi(x)$$

$$P.I = \frac{1}{f(D)} x^m \sin ax$$

$$P.I = \text{Imaginary part of (I.P) } \frac{1}{f(D)} x^m (\cos ax + i \sin ax)$$

$$P.I = \text{I.P of } \frac{1}{f(D)} x^m e^{iax}$$

this can be Evaluated.

Put $\phi(x) = x^m \cos ax$:-

$$P.I = \frac{1}{f(D)} \phi(x) = \frac{1}{f(D)} x^m \cos ax$$

$$= \text{Real part of (R.P) } \frac{1}{f(D)} x^m (\cos ax + i \sin ax)$$

$$= \text{R.P of } \frac{1}{f(D)} x^m e^{iax}$$

this can be Evaluated.

Note :- If $\phi(x) = x.V$ (when $m=1$) where 'V' is a function of 'x'.

then P.I is
$$P.I = \frac{1}{f(D)}\phi(x)$$
$$= \frac{1}{f(D)} x.V.$$
$$P.I = \left[x - \frac{1}{f(D)} f'(D)\right]\frac{1}{f(D)}.V.$$

Problems :- ① Solve $(D^2+4)y = x\sin x$

Sol: $y_c = c_1\cos 2x + c_2\sin 2x$ — (1)

$$y_p = \frac{1}{D^2+4} x\sin x$$
$$= \text{I.P of } \frac{1}{D^2+4} x e^{ix}$$
$$= \text{I.P of } e^{ix}\frac{1}{(D+i)^2+4}.x$$
$$= \text{I.P of } e^{ix}\frac{1}{D^2+2Di+3}.x$$
$$= \text{I.P of } \frac{e^{ix}}{3}\left[1+\left(\frac{D^2+2Di}{3}\right)\right]^{-1}x$$
$$= \text{I.P of } \frac{e^{ix}}{3}\left[1-\left(\frac{D^2+2Di}{3}\right)\right]x$$
$$= \text{I.P of } \frac{e^{ix}}{3}\left[x - \frac{2i}{3}(1)\right]$$
$$= \text{I.P of } \frac{e^{ix}}{3}\left(x-\frac{2}{3}i\right) = \text{I.P of } \frac{(\cos x + i\sin x)}{3}\left(x-\frac{2}{3}i\right)$$
$$= \text{I.P of } \frac{1}{3}\left[\cancel{x\cos x} - \frac{2}{3}i\cos x + xi\sin x + \frac{2}{3}\sin x\right]$$
$$y_p = \frac{1}{3}\left(-\frac{2}{3}\cos x + x\sin x\right) \text{ — (2)}$$

$\boxed{\therefore y = y_c + y_p}$

(2). Solve $\frac{d^2y}{dx^2} - y = x\sin x + (1+x^2)e^x$.

Sol: $(D^2-1)y = x\sin x + (1+x^2)e^x$.

$y_c = c_1e^x + c_2e^{-x}$.

$y_p = \underbrace{\frac{1}{D^2-1}x\sin x}_{(y_{p_1})} + \underbrace{\frac{1}{D^2-1}(1+x^2)e^x}_{(y_{p_2})}$ ——— (1)

$y_{p_1} = \text{I.P of } \frac{1}{D^2-1} x e^{ix}$

$= \text{I.P of } e^{ix}\frac{1}{(D+i)^2-1}\cdot x \quad = \text{I.P of } e^{ix}\frac{1}{D^2+2Di-2}\cdot x$

$= \text{I.P of } e^{ix}\frac{1}{-2\left[1-\left(\frac{D^2+2Di}{2}\right)\right]}\cdot x$

$= \text{I.P of } \frac{e^{ix}}{-2}\left[1-\left(\frac{D^2+2Di}{2}\right)\right]^{-1} x$

$= \text{I.P of } (\cos x + i\sin x)\left(-\frac{1}{2}\right)\left[1+\left(\frac{D^2+2Di}{2}\right)\right] x$

$= \text{I.P of } (\cos x + i\sin x)\left(-\frac{1}{2}\right)\left(x+\frac{2i}{2}\right)$

$y_{p_1} = -\frac{1}{2}(x\sin x + \cos x)$ ——— (2).

$y_{p_2} = \frac{1}{D^2-1}(1+x^2)e^x$

$= e^x\frac{1}{(D+1)^2-1}(1+x^2) = e^x\frac{1}{D^2+2D}(1+x^2)$

$= e^x\frac{1}{2D\left[\frac{D+2}{2}\right]}(1+x^2)$

$$= \frac{e^x}{2D}\left(1+\frac{D}{2}\right)^{-1}(1+x^2)$$

$$= \frac{e^x}{2D}\left(1-\frac{D}{2}+\frac{D^2}{4}\right)(1+x^2)$$

$$= \frac{e^x}{2D}\left(1+x^2-\frac{D}{2}-\frac{D}{2}x^2+\frac{D^2}{4}+\frac{D^2}{4}x^2\right)$$

$$= \frac{e^x}{2D}\left(1+x^2-\frac{D}{2}-x+\frac{D^2}{4}+\frac{1}{2}\right)$$

$$= e^x\left(\frac{1}{2D}+\frac{1}{2D}x^2-\frac{1}{4}-\frac{x}{2D}+\frac{D}{8}+\frac{1}{4D}\right)$$

$$= e^x\left(\frac{1}{2}x+\frac{1}{2}\frac{x^3}{3}-\frac{1}{4}-\frac{1}{2}\frac{x^2}{2}+0+\frac{1}{4}x\right)$$

$$= e^x\left(\frac{x^3}{6}-\frac{x^2}{4}+\frac{3}{4}x-\frac{1}{4}\right)$$

$$y_{p_2} = \frac{e^x}{2}\left(\frac{x^3}{3}-\frac{x^2}{2}+\frac{3}{2}x-\frac{1}{2}\right) \text{ ——— } (3).$$

sub. (2) & (3) in (1).

$$y_p = -\frac{1}{2}(x\sin x+\cos x)+\frac{e^x}{2}\left(\frac{x^3}{3}-\frac{x^2}{2}+\frac{3}{2}x-\frac{1}{2}\right)$$

$$\therefore \boxed{y = y_c+y_p}.$$

(3). Solve the D.E $(D^4+2D^2+1)y = x^2\cos^2 x$.

Sol^n:- $\boxed{y = y_c+y_p}$ — (1)

y_c:- A.E is $f(m)=0 \Rightarrow m^4+2m^2+1=0 \Rightarrow (m^2+1)^2=0$

$m = \pm i, \pm i$

$m = +i, +i, -i, -i$

$m = \alpha \pm i\beta, \alpha \pm i\beta$.

Here $\alpha \pm i\beta$ repeated twice

$\alpha=0, \beta=1$.

$$y_c = \left[(c_1+c_2x)\cos\beta x+(c_3+c_4x)\sin\beta x\right]e^{\alpha x}$$

$$y_c = (c_1+c_2x)\cos x+(c_3+c_4x)\sin x \text{ —— } (2)$$

$$y_p = \frac{1}{f(D)}\phi(x) = \frac{1}{D^4+2D^2+1} x^2\cos^2 x$$

$$= \frac{1}{D^4+2D^2+1} \frac{x^2(1+\cos 2x)}{2} \quad \left(\because \cos 2\theta = 2\cos^2\theta - 1\right)$$

$$= \frac{1}{2(D^4+2D^2+1)} x^2(1+\cos 2x)$$

$$= \frac{x^2}{2(D^2+1)^2} + \frac{x^2\cos 2x}{2(D^4+2D^2+1)}$$

$$\boxed{y_p = y_{p_1} + y_{p_2}} \quad \text{--- (3)}$$

where $y_{p_1} = \frac{x^2}{2(D^2+1)^2}$

$$= \frac{x^2}{2}(1+D^2)^{-2}$$

$$= \frac{x^2}{2}(1-2D^2) \quad \left(\because (1+D)^{-2} = 1-2D+3D^2-4D^3+ \cdots\right)$$

$$= \frac{x^2}{2} - 2. \quad \text{--- (4)}$$

where $y_{p_2} = \frac{x^2\cos 2x}{2(D^4+2D^2+1)}$

$$= \text{R.P of } \frac{1}{2(D^4+2D^2+1)} x^2 e^{i2x} = \text{R.P of } \frac{1}{2(D^2+1)^2} x^2 e^{i2x}$$

$$= \text{R.P of } \frac{e^{i2x}}{2} \frac{1}{\left[(D+2i)^2+1\right]^2} x^2$$

$$= \text{R.P of } \frac{e^{i2x}}{2} \frac{x^2}{(D^2+4i^2+4Di+1)^2}$$

$$= \text{R.P of } \frac{e^{i2x}}{2} \frac{x^2}{(D^2+4Di-3)^2}$$

$$= R.P \text{ of } \frac{e^{i2x}}{2} \frac{x^2}{(-3)^2\left[1-\left(\frac{D^2+4Di}{3}\right)\right]^2}$$

$$= R.P \text{ of } \frac{e^{i2x}}{18} x^2\left[1-\left(\frac{D^2+4Di}{3}\right)\right]^{-2} \qquad \left(\because (1-D)^{-2} = 1+2D+3D^2+\dots\right)$$

$$= R.P \text{ of } \frac{e^{i2x}}{18}\left[1+2\left(\frac{D^2+4Di}{3}\right)+3\left(\frac{D^2+4Di}{3}\right)^2\right]x^2$$

$$= R.P \text{ of } \frac{e^{i2x}}{18}\left[1+\frac{2D^2}{3}+\frac{8Di}{3}+\frac{3(16D^2i^2)}{9}\right]x^2$$

$$= R.P \text{ of } \frac{e^{i2x}}{18}\left[x^2+\frac{2}{3}(2)+\frac{8}{3}i(2x)+\frac{16}{3}(-1)(2)\right]$$

$$= R.P \text{ of } \frac{e^{i2x}}{18}\left(x^2+\frac{4}{3}+\frac{16xi}{3}-\frac{32}{3}\right)$$

$$= R.P \text{ of } \left(\frac{\cos 2x + i\sin 2x}{18}\right)\left(x^2+\frac{4}{3}+\frac{16ix}{3}-\frac{32}{3}\right)$$

$$= \frac{1}{18}\left(x^2\cos 2x+\frac{4}{3}\cos 2x-\frac{32}{3}\cos 2x-\frac{16}{3}x\sin 2x\right)$$

$$y_{p_2} = \frac{1}{18}\left(x^2\cos 2x-\frac{28}{3}\cos 2x-\frac{16x}{3}\sin 2x\right) \quad \text{--- (5)}$$

sub,, (4) & (5) in (3)

$$\boxed{y_p = y_{p_1}+y_{p_2}} \quad \text{--- (6)}.$$

sub,, (2) & (6) in (1).

$$\therefore \boxed{y = y_c + y_p}$$

//

H.W

(4). $(D^2-4D+4)y = x^2\sin x + e^{2x}+3.$

[A:- $y = (c_1+c_2x)e^{2x} + \frac{1}{625}\left[(220x+244)\cos x + (40x+33)\sin x\right] + \frac{x^2}{2}e^{2x} + \frac{3}{4}$]

H.W

(5). Solve $(D^2+1)y = x^2\sin 2x$

[A:- $-\frac{1}{3}\left(x^2\sin 2x + \frac{8x}{3}\cos 2x - \frac{26}{9}\sin 2x\right)$].

(6). Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = xe^x\sin x$

[A:- $y = c_1e^{-x} + c_2e^{-2x} + e^x\left[\frac{x}{10}(\sin x - \cos x) - \frac{1}{25}\sin x + \frac{1}{10}\cos x\right]$]

(7) Solve $(D^2+1)x = t\cos t$ given $x=0,\ \frac{dx}{dt}=0$ at $t=0$

Sol:- A.E is $m^2+1=0 \Rightarrow m = \pm i$

$C.F = x_c = c_1\cos t + c_2\sin t$ —— (i).

$P.I = x_p = \frac{1}{f(D)}\cdot Q(t)$

$= \frac{1}{D^2+1}\, t\cos t$

$= R.P \text{ of } \frac{1}{D^2+1}\, t\, e^{it}$

$= R.P \text{ of } e^{it}\frac{1}{(D+i)^2+1}\, t.$

$= R.P \text{ of } e^{it}\frac{1}{(D^2+2Di)}\, t$

$= R.P \text{ of } e^{it}\frac{1}{2Di\left(1+\frac{D^2}{2Di}\right)}\, t$

$= R.P \text{ of } \frac{e^{it}}{2Di}\left(1+\frac{D}{2i}\right)^{-1} t.$

$$= R.P \text{ of } \frac{e^{it}}{2Di}\left(1-\frac{D}{2i}\right)t$$

$$= R.P \text{ of } \frac{e^{it}}{2Di}\left(t-\frac{1}{2i}\right)$$

$$= R.P \text{ of } \frac{e^{it}}{2i}\left(\frac{t^2}{2}-\frac{1}{2i}t\right)$$

$$= R.P \text{ of } \left(\frac{\cos t + i\sin t}{2i}\right)\left(\frac{t^2}{2}-\frac{1}{2i}t\right)$$

$$= R.P \text{ of } \left(\frac{\cos t . t^2}{4i} - \frac{1}{4i^2}t\cos t + \frac{i\sin t}{2i}\frac{t^2}{2} - \frac{i\sin t . t}{4i^2}\right)$$

$$= R.P \text{ of } \left(\frac{\cos t . t^2}{4i} + \frac{1}{4}t\cos t + \frac{\sin t}{2}\frac{t^2}{2} + \frac{it\sin t}{4}\right)$$

$$x_p = \frac{1}{4}t\cos t + \frac{t^2}{4}\sin t \quad \text{---- (2)}$$

$\therefore \; x = x_c + x_p$

$$\boxed{x = c_1\cos t + c_2\sin t + \frac{1}{4}t\cos t + \frac{t^2}{4}\sin t}$$

Given $x=0$ at $t=0$

$0 = c_1(1) + 0 + 0 + 0.$

$\boxed{c_1 = 0}$.

Given $\frac{dx}{dt} = 0$ at $t=0$

$$c_1(-\sin t) + c_2(\cos t) + \frac{1}{4}\left[t(-\sin t) + \cos t\right] + \frac{1}{4}\left[t^2\cos t + \sin t(2t)\right] = 0$$

$$\Rightarrow 0 + c_2(1) + \frac{1}{4}[0+1] + \frac{1}{4}(0+0) = 0 \Rightarrow \boxed{c_2 = -1/4}$$

$\therefore$ G.S is $x = -\frac{1}{4}\sin t + \frac{1}{4}t\cos t + \frac{t^2}{4}\sin t$ //

Variation of Parameters :-

Working Rule :-

(1). Reduce the given Eqn of the form $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = R$

(2). Find C.F $= y_c = c_1u + c_2v$. where u & v are functions of 'x'.

(3). Take P.I. $y_p = Au + Bv$, where A & B are functions of 'x'.

$$A = -\int \frac{vR\,dx}{uv' - vu'} \quad \text{and} \quad B = \int \frac{uR\,dx}{uv' - vu'}$$

(4). write the G.S of the given Equation $y = y_c + y_p$.

Problems :-

(1). Apply the method of variation of parameters to solve

$\frac{d^2y}{dx^2} + y = \operatorname{cosec} x$.

Sol:- $(D^2+1)y = \operatorname{cosec} x$,

A.E is $m^2+1=0 \Rightarrow m = \pm i$

$y_c = c_1\cos x + c_2\sin x$ ——— (1)

$(y_c = c_1 u + c_2 v)$

where $u = \cos x$; $v = \sin x$.

$\boxed{y_p = Au + Bv}$ ——— (2)

where $A = -\int \frac{\sin x \cdot \operatorname{cosec} x\, dx}{\cos x(\cos x) + \sin x(\sin x)}$; $B = \int \frac{\cos x \cdot \operatorname{cosec} x\, dx}{1}$

$= -\int \frac{1}{1}\, dx$

$\boxed{A = -x}$

$B = \int \cot x\, dx$

$\boxed{B = \log(\sin x)}$

From (2) $\Rightarrow y_p = xu + \log(\sin x)v$

$$\boxed{y_p = -x\cos x + \log(\sin x)\sin x} \quad \text{---} (3)$$

$\therefore$ G.S is $y = y_c + y_p$

$$y = c_1\cos x + c_2\sin x + x\cos x + \log(\sin x)\sin x$$

(2). Solve $(D^2+a^2)y = \tan ax$ HW $(D^2+4)y = \tan 2x$.

Sol: A.E is $f(m) = 0$

$$m^2 + a^2 = 0 \Rightarrow m = \pm ai$$

$$\therefore \boxed{y_c = c_1\cos ax + c_2\sin ax} \quad \text{---} (1)$$

$(y_c = c_1 u + c_2 v)$

where $u = \cos ax$; $v = \sin ax$

$$\boxed{y_p = Au + Bv} \quad \text{---} (2)$$

where $A = -\int \frac{vR}{uv' - vu'}dx = -\int \frac{\sin ax . \tan ax}{a\cos ax\cos ax + a\sin ax\sin ax} dx$

$$= -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax} dx$$

$$= -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax} dx$$

$$= -\frac{1}{a}\int [\sec ax - \cos ax] dx$$

$$= -\frac{1}{a^2}\left[\log(\sec ax + \tan ax) - \sin ax\right]$$

$$A = -\frac{1}{a^2}\log(\sec ax + \tan ax) + \frac{1}{a^2}\sin ax$$

$$B = \int \frac{uR\,dx}{a} = \int \frac{\cos ax \tan ax}{a}\,dx = \frac{1}{a}\left(\frac{-\cos ax}{a}\right)$$

$$= -\frac{1}{a^2}\cos ax.$$

$\therefore y_p = Au + Bv$

$$y_p = \cos ax\left[-\frac{1}{a^2}\log(\sec ax + \tan ax) + \frac{1}{a^2}\sin ax\right] - \frac{1}{a^2}\cos ax(\sin ax)$$

$$= -\frac{1}{a^2}\cos ax.\log(\sec ax + \tan ax) + \cancel{\frac{1}{a^2}\sin ax\cos ax} - \cancel{\frac{1}{a^2}\cos ax(\sin ax)}$$

$$y_p = -\frac{1}{a^2}\cos ax.\log(\sec ax + \tan ax) \quad \text{--- (3)}$$

$\therefore y = y_c + y_p$

$$= c_1\cos ax + c_2\sin ax - \frac{\cos ax}{a^2}\log(\sec ax + \tan ax)$$

//.

(3). $(D^2 + a^2)y = \sec ax$.

Sol: $y_c = c_1\cos ax + c_2\sin ax$ --- (1)

$(y_c = c_1 u + c_2 v)$

$y_p = Au + Bv$. --- (2)

$$A = -\int \frac{vR\,dx}{uv' - vu'} = -\int \frac{\sin ax.\sec ax}{a\cos ax\cos ax + a\sin ax\sin ax}\,dx$$

$$= -\int \frac{\tan ax}{a}\,dx = -\frac{1}{a}\frac{\log(\sec ax)}{a}.$$

$$= -\frac{1}{a^2}\log(\sec ax)$$

$$B = +\int \frac{uR\,dx}{uv' - vu'} = +\int \frac{\cos ax.\sec ax}{a}\,dx$$

$$= +\int \frac{1\,dx}{a} = +x/a.$$

Sub. A & B in (2), we get 'y_p' then sub. in $\boxed{y_c + y_p = y}$ //

4. solve $y''-6y'+9y=\frac{e^{3x}}{x^2}$.

Sol: y_c: A.E is $m^2-6m+9=0$.

$$m^2-3m-3m+9=0$$

$$m(m-3)-3(m-3)=0$$

$$m=3,3.$$

$\therefore y_c=(c_1+c_2x)e^{3x}$ —— (1) $\Rightarrow c_1e^{3x}+c_2xe^{3x}=y_c$

$(c_1u+c_2v=y_c)$

$y_p=Au+Bv$ —— (2)

$$A=-\int\frac{vR\,dx}{uv'-uv'}=-\int\frac{\cancel{x}e^{3x}\cdot\frac{e^{3x}}{x\cancel{x}}\,dx}{e^{3x}\left(xe^{3x}(3)+e^{3x}\right)-xe^{3x}e^{3x}(3)}$$

$$=-\int\frac{\frac{e^{6x}}{x}\,dx}{\cancel{3xe^{6x}}+e^{6x}-\cancel{3xe^{6x}}}$$

$$=-\int\frac{\cancel{e^{6x}}}{x}\cdot\frac{1}{\cancel{e^{6x}}}\,dx=\int 1/x\,dx=-\log x+c.$$

$\therefore \boxed{A=-\log x}$

$$B=\int\frac{uR\,dx}{uv'-vu'}=\int\frac{e^{3x}\cdot\frac{e^{3x}}{x^2}\,dx}{e^{6x}}=\int\frac{1}{x^2}\,dx$$

$$=\int x^{-2}\,dx=\frac{x^{-2+1}}{-2+1}=\frac{x^{-1}}{-1}=-1/x$$

subn A & B in (2) we get 'y_p'.

Subn y_c & y_p in

$\boxed{y=y_c+y_p}$

5. $(D^2-2D)y = e^x \sin x$

Sol: $y_c = c_1 + c_2 e^{2x}$ — (1)

$(y_c = c_1 u + c_2 v)$

$y_p = Au + Bv$ — (2)

$$A = -\int \frac{vR\,dx}{uv' - vu'} = -\int \frac{e^{2x} \cdot e^x \sin x\,dx}{2e^{2x} - e^{2x}(0)} = -\frac{1}{2}\int e^x \sin x\,dx$$

$$= -\frac{1}{2}\left[\frac{e^x}{1+1}(\sin x - \cos x)\right]$$

$$= -\frac{1}{4} e^x (\sin x - \cos x)$$

$$B = \int \frac{uR\,dx}{uv' - vu'} = \int \frac{1 \cdot e^x \sin x}{2e^{2x}} = \frac{1}{2}\int e^{-x} \sin x = \frac{1}{2}\left[\frac{e^{-x}}{1+1}(-\sin x - \cos x)\right]$$

$$= -\frac{1}{4} e^{-x} (\sin x + \cos x)$$

∴ Sub" A & B in (2), we get 'y_p'.

G.S is $\boxed{y = y_c + y_p}$.

6. $(D^2+4)y = \sec 2x$ $\left(y = c_1\cos 2x + c_2 \sin 2x + \frac{\cos 2x}{4}\log|\cos 2x| + \frac{x}{2}\sin 2x\right)$

7. $(D^2+1)y = \cos x$ $\left(\because\ y = c_1\cos x + c_2 \sin x + \frac{1}{4}\cos x \cdot \cos 2x + \frac{1}{2}\sin x\,(x + \sin x \cos x)\right)$

8. $(D^2+1)y = x\cos x$.

Sol: $y_c = c_1 \cos x + c_2 \sin x$ — (1)

$y_p = Au + Bv$ — (2)

$$A = -\int \frac{vR\,dx}{uv' - vu'} = -\int \frac{\sin x \cdot x\cos x}{1} = -\frac{1}{2}\int x \sin 2x\,dx$$

$$= -\frac{1}{2}\left[\frac{-x\cos 2x}{2} - \int -\frac{\cos 2x}{2}\,dx\right] = \frac{x\cos 2x}{4} - \frac{\sin 2x}{8}$$

$B = \int \frac{w R\, dx}{w' - v w} = \int \frac{\cos x . x\cos x}{1} dx = \frac{1}{1}\int x\cos^2 x\, dx$

$= \int x\left(\frac{1+\cos 2x}{2}\right) dx \quad (\because \cos 2\theta = 2\cos^2\theta - 1)$

$= \left[x\left(\frac{x}{2}+\frac{\sin 2x}{4}\right) - \int\left(\frac{x}{2}+\frac{\sin 2x}{4}\right) dx\right]$

$= \frac{x^2}{2} + \frac{x\sin 2x}{4} - \frac{x^2}{4} + \frac{\cos 2x}{8}$

$B. = \frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8}$

sub. A & B in (2) we get 'y_p'.

$\therefore$ G.S is $\boxed{y = y_c + y_p}$.

(9). $(D^2+4)y = x\sin x$

$y = c_1\cos x + c_2\sin x - \left\{\frac{x^2}{2} - \frac{x\sin 2x}{4} - \frac{x^2}{8} - \frac{\cos 2x}{8}\right\}\cos x +$

$\left\{\frac{x}{4}\cos 2x + \frac{1}{8}\sin 2x\right\}\sin x$ //

Homogeneous Linear D.E (or) Cauchy Euler linear D.E :-

An Eqⁿ of the form $x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots +$

$$a_{n-1} x^1 \frac{dy}{dx} + a_n y = \phi(x) \quad \text{— (1)}$$

where $a_1, a_2, \dots a_n$ are real constants and $\phi(x)$ is a function of 'x' is called a H.L.D.E (or) cauchy - Euler - L.D.E of order 'n'.

Eqⁿ(1) can be written in the operator form as

$$\left[x^n D^n + a_1 x^{n-1} D^{n-1} + a_2 x^{n-1} D^{n-2} + \dots + a_{n-1} x D + a_n\right] y = \phi(x)$$

where $D = \frac{d}{dx}$.

Cauchy's L.D.E can be transformed into a L.D.E with constant coefficients by change of independent variable with the substitution :

$$x = e^z \text{ (or) } z = \log x.$$

Diff. w.r.t 'x'.

$$\frac{dz}{dx} = \frac{1}{x},$$

Consider $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{1}{x}$$

$$Dy = \theta . y . \frac{1}{x} \qquad \text{where } \theta = \frac{d}{dz}$$

$$\Rightarrow \boxed{xD = \theta}$$

& $x^2D^2 = \theta(\theta-1)$

$x^3D^3 = \theta(\theta-1)(\theta-2)$.

- - - - - - -

$x^nD^n = \theta(\theta-1)(\theta-2)\cdots\cdots(\theta-(n-1))$,

1. Solve $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = \log x$.

The given D.E is of the form Cauchy - Euler L.D.E and the operator form of given D.E

$$\left(x^2D^2 - xD + 1\right)y = \log x. \quad \text{—} (1)$$

let $x = e^z$ (or) $z = \log x$.

Take $xD = \theta$

$x^2D^2 = \theta(\theta-1)$

Eqn (1) becomes $\Rightarrow \left[\theta(\theta-1) - \theta + 1\right]y = z$

$\Rightarrow (\theta^2 - 2\theta + 1)y = z$

$\Rightarrow f(\theta)y = z$.

G.S is : $\boxed{y = y_c + y_p}$ — (2)

$y_c = (c_1 + c_2 z)e^{mz} = (c_1 + c_2 z)e^{z}$

$P.I = y_p = \dfrac{1}{(\theta^2 - 2\theta + 1)}z$

$= \left[1 + (\theta^2 - 2\theta)\right]^{-1} z \qquad \left(\because \theta = \dfrac{d}{dz}\right)$

$y_p = \left[1 - (\theta^2 - 2\theta)\right]z = z - \theta^2 z + 2\theta z = z + 2(1) = z + 2$.

$\therefore \boxed{y = y_c + y_p}$.

$\therefore y = (C_1 + C_2 z)e^{z} + z + 2$

$= [C_1 + C_2(\log x)]x + \log x + 2$ //

(2). Solve. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = (1+x)^2$

Sol: $(x^2D^2y - 3xDy + 4y) = (1+x)^2$

$\Rightarrow x^2D^2y - 3xDy + 4y = 1 + x^2 + 2x$

$\Rightarrow (x^2D^2 - 3xD + 4)y = 1 + x^2 + 2x$ —— (1)

put $z = \log x$ (or) $x = e^{z}$

& $xD = \theta$

$x^2D^2 = \theta(\theta - 1)$

$\Rightarrow [\theta(\theta-1) - 3\theta + 4]y = 1 + e^{2z} + 2e^{z}$

$\Rightarrow (\theta^2 - 4\theta + 4)y = 1 + e^{2z} + 2e^{z}$

$f(\theta)y = \phi(z)$.

$\therefore \boxed{y = y_c + y_p}$ —— (2)

$y_c = (C_1 + C_2 z)e^{2z}$ —— (3).

$y_p = \dfrac{1}{\theta^2 - 4\theta + 4}(1 + e^{2z} + 2e^{z})$

$= \dfrac{e^{0.z}}{\theta^2 - 4\theta + 4} + \dfrac{e^{2z}}{\theta^2 - 4\theta + 4} + \dfrac{2e^{z}}{\theta^2 - 4\theta + 4}$

$y_p = \dfrac{1}{4} + \dfrac{e^{2z}}{(\theta-2)^2} + \dfrac{2e^{z}}{1} = \dfrac{1}{4} + \dfrac{z^2 e^{2z}}{2!} + 2e^{z}$

$\therefore y = (C_1 + C_2\log x)e^{2\log x} + \dfrac{1}{4} + \dfrac{(\log x)^2 \cdot x^2}{2} + 2x$ //

3. Solve $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\frac{12\log x}{x}$.

Sol: $\frac{d^2y}{dx^2}+\frac{1}{x}\frac{dy}{dx}=\frac{12\log x}{x}$

multiply with given 'x^2'

$\Rightarrow$ $x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}=12x\log x$

$\Rightarrow (x^2D^2+xD)y=12x\log x$

$f(D)y=\phi(x)$.

put $e^z=x \Rightarrow z=\log x$

$xD=\theta$

$x^2D^2=\theta(\theta-1)$

$\Rightarrow (\theta^2-\theta+\theta)y=12ze^z$

$f(\theta)y=\phi(z)$

$\therefore \boxed{y=y_c+y_p}$ — (1)

$y_c=(c_1+c_2z)e^{0.z}$

$y_c=c_1+c_2z$ — (2)

$y_p=\frac{1}{f(\theta)}\cdot\phi(z)$

$=\frac{1}{(\theta^2-\theta+\theta)}12ze^z$

$=\frac{1}{\theta^2}12ze^z$

$=12\left(\frac{1}{\theta^2}ze^z\right)=12e^z\left(\frac{1}{(\theta+1)^2}\cdot z\right)$

$=12e^z(1+\theta)^{-2}z = 12e^z(1-2\theta)z$

$y_p=12e^z(z-2)=12ze^z-24e^z$

$\therefore y=(c_1+c_2\log x)+12x(\log x-2)$

④. $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x\log x$.

⑤. $(x^2D^2 - 2xD - 4)y = x^2 \cdot 2\log x$.

Sol:- Put $xD = \theta$ & $z = \log x$ & $x = e^z$

$x^2D^2 = \theta(\theta-1)$

$\Rightarrow (\theta(\theta-1) - 2\theta - 4)y = (e^z)^2 \cdot 2z$

$\Rightarrow (\theta^2 - 3\theta - 4)y = 2e^{2z}z$

$f(\theta)y = \phi(z)$.

where $f(\theta) = \theta^2 - 3\theta - 4$

$\phi(z) = 2ze^{2z}$

$\boxed{y = y_c + y_p}$ —(1)

$\Rightarrow$ y_c :- A.E of $f(m) = 0$ $\Rightarrow m^2 - 3m - 4 = 0$

$\Rightarrow m^2 - 4m + m - 4 = 0$

$\Rightarrow m(m-4) + 1(m-4) = 0$

$(m-4)(m+1) = 0$

$m = -1, 4$.

$\therefore \boxed{y_c = c_1e^{-z} + c_2e^{4z}}$ —(2)

$\Rightarrow$ y_p :- $y_p = \frac{1}{f(\theta)}\phi(z)$

$= \frac{1}{\theta^2 - 3\theta - 4} 2ze^{2z}$

$= 2e^{2z}\frac{1}{(\theta+2)^2 - 3(\theta+2) - 4} z$

$= 2e^{2z}\frac{1}{\theta^2 + 4\theta + \cancel{4} - 3\theta - 6 - \cancel{4}} \cdot z$

$= 2e^{2z}\frac{1}{\theta^2 + \theta - 6} z$

$= 2e^{2z}\frac{1}{-6\left[1 - \left(\frac{\theta^2 + \theta}{6}\right)\right]} z$

$$= \frac{2e^{2z}}{-6}\left[1-\left(\frac{\theta^2+\theta}{6}\right)\right]^{-1} \cdot z$$

$$= -\frac{1}{3}e^{2z}\left[1+\left(\frac{\theta^2+\theta}{6}\right)\right] z$$

$$= -\frac{1}{3}e^{2z}\left(1+\frac{\theta^2}{6}z+\frac{\theta}{6}z\right)$$

$$= -\frac{1}{3}e^{2z}\left(1+0+\frac{1}{6}\right)$$

$$y_p = -\frac{1}{3}e^{2z}\left(\frac{7}{6}\right) = -\frac{7}{18}e^{2z} \quad \text{—— (3)}$$

subs (2) & (3) in (1)

$\boxed{y = y_c + y_p}$ (∵ put $z = \log x$ & $x = e^z$)

(6). $x^4\frac{d^3y}{dx^3} + 2x^3\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} + xy = 1$

Sol:- the given D.E can be written in operator form as

$[x^4D^3 + 2x^3D^2 - x^2D + x]\, y = 1$ and the G.S is $\boxed{y = y_c + y_p}$

Divide by 'x' on b.s.

$[x^3D^3 + 2x^2D^2 + 1 - xD]\, y = 1/x$

$\Rightarrow (x^3D^3 + 2x^2D^2 - xD + 1)\, y = 1/x$. —— (1)

and let $x = e^z \Rightarrow z = \log x$

$xD = \theta$

$x^2D^2 = \theta(\theta-1)$

$x^3D^3 = \theta(\theta-1)(\theta-2)$

(1) $\Rightarrow (\theta^3 - \theta^2 - \theta + 1)\, y = e^{-z}$

$f(\theta)y = Q(z)$

$f(\theta) = \theta^3 - \theta^2 - \theta - 1$

$Q(z) = e^{-z}$

$\underline{y_c}$:- $y_c = (c_1+c_2 z)e^{z} + c_3 e^{-z}$ ——(2) $(\because m = 1, 1, -1)$

$y_p = \frac{1}{f(\theta)} \phi(z) = \frac{1}{\theta^3-\theta^2-\theta+1} e^{-z}$

$= \frac{1}{(\theta-1)(\theta^2-1)} e^{-z}$

$= \frac{1}{(\theta-1)(\theta+1)(\theta-1)} e^{-z}$ (Put $\theta = -1$)

$= \frac{1}{(-2)(-2)(\theta+1)} e^{-z}$

$= \frac{1}{4} e^{-z} \cdot \frac{z^1}{1!}$

$y_p = \frac{z e^{-z}}{4}$ ——(3)

Sub,, (2) & (3) in (1) $\boxed{\therefore y = y_c + y_p}$ & sub,, $x = e^z$ and $z = \log x$.

⑦. $x^3D^3y + 2x^2D^2y + 2y = 10(x + 1/x)$ ——(1)

Soln: Put $x = e^z$

$\Rightarrow z = \log x$

Put $xD = \theta$; $x^2D^2 = \theta(\theta-1)$; $x^3D^3 = \theta(\theta-1)(\theta-2)$

(1) $\Rightarrow (\theta^3 - \theta^2 + 2)y = 10(e^z + e^{-z})$ ——(2).

$f(\theta)y = \phi(z)$

where $f(\theta) = \theta^3 - \theta^2 + 2$; $\phi(z) = 10(e^z + e^{-z})$

G.S is $\boxed{y = y_c + y_p}$

$\underline{y_c}$:- $m = -1, 1 \pm i$

$y_c = c_1 e^{-z} + (c_2 \cos z + c_3 \sin z)e^z$.

$y_c = c_1 \cdot \frac{1}{x} + (c_2 \cos \log x + c_3 \sin \log x)x$.

——(3).

$$y_p = \frac{1}{f(\theta)} \phi(z)$$

$$= \frac{1}{\theta^3 - \theta^2 + 2} \; 10(e^z + e^{-z})$$

$$= 5e^z + \frac{z \, 10 e^{-z}}{1! \, (1+2+2)}$$

$$= 5e^z + 2e^{-z} \cdot z$$

$$y_p = 5x + 2 \log x \, (1/x) \quad \text{--- (4).}$$

$$\therefore \boxed{y = y_c + y_p}$$

(8). Solve $x^3 D^3 y + 3x^2 D^2 y + x D y + 8y = 65 \cos(\log x)$.

Sol: $x^3 D^3 y + 3x^2 D^2 y + x D y + 8y = 65(\cos(\log x))$ --- (1)

put $x = e^z \Rightarrow \log x = z$

& $xD = \theta$

$x^2 D^2 = \theta(\theta - 1)$

$x^3 D^3 = \theta(\theta-1)(\theta-2)$

$(1) \Rightarrow (\theta^3 + 8) y = 65 \cos z$ --- (2)

$\boxed{y = y_c + y_p}$ --- (3)

$y_c = c_1 e^{-2z} + (c_2 \cos\sqrt{3} z + c_3 \sin\sqrt{3} z) e^z$ $\quad (m = -2, \; m = 1 \pm \sqrt{3} i)$

$\Rightarrow$

$$y_c = c_1 \frac{1}{x^2} + \left[c_2 \cos(\sqrt{3} \log x) + c_3 \sin(\sqrt{3} \log x)\right] x \quad \text{--- (4).}$$

$$\Rightarrow y_p = \frac{1}{\theta^3 + 8} \, 65 \cos z$$

$$= \frac{1}{\theta^2 \cdot \theta + 8} \, 65 \cos z \quad (\text{put } \theta^2 = -1)$$

$$= \left(\frac{1}{8 - \theta} \times \frac{8 + \theta}{8 + \theta}\right) 65 \cos z$$

$\Rightarrow \quad y_p = (8+\theta)\cos z$

$= 8\cos z + \theta \cos z$

$= 8\cos z - \sin z \qquad \left(\because \theta = \frac{d}{dz}\right)$

$y_p = 8\cos(\log x) - \sin(\log x) \quad \text{—— } 5)$

Sub. (4) & (5) in (3) ie, $\boxed{y = y_c + y_p}$

9. $(x^2D^2 - 3xD + 1)\,y = \dfrac{\log x \,\sin(\log x) + 1}{x} \quad \text{—— } (1)$

Sol: $(\theta^2 - \theta - 3\theta + 1)\,y = \dfrac{z \cdot \sin z + 1}{e^z}$

$\Rightarrow \qquad = z \sin z\, e^{-z} + e^{-z}$

$\boxed{f(\theta)\,y = Q(z)}$

$\boxed{y = y_c + y_p} \quad \text{—— } (2)$

$\Rightarrow$ y_c :- $m = 2 \pm \sqrt{3}. \ (\alpha \pm \sqrt{\beta})$

$\therefore y_c = (c_1 \cosh\sqrt{\beta}x + c_2 \sinh\sqrt{\beta}x)\,e^{\alpha x}$

$y_c = e^{2x}(c_1 \cosh\sqrt{3}x + c_2 \sinh\sqrt{3}x) \quad \text{—— } (3).$

$\Rightarrow$ $y_p = \dfrac{1}{\theta^2 - 4\theta + 1}\left(z e^{-z} \sin z + e^{-z}\right)$

$y_p = \dfrac{z e^{-z} \sin z}{\theta^2 - 4\theta + 1} + \dfrac{e^{-z}}{\theta^2 - 4\theta + 1}$

$\boxed{y_p = y_{p_1} + y_{p_2}}$

Here $y_{p_2} = \dfrac{e^{-z}}{\theta^2 - 4\theta + 1}$ put $\theta = -1 \Rightarrow y_{p_2} = \dfrac{e^{-z}}{1+4+1} = \dfrac{e^{-z}}{6} = \dfrac{1}{6x}$

$\left(\because e^{-z} = 1/x\right)$

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Now $y_{p_1} = \dfrac{e^{-z} \cdot z \sin z}{D^2 - 4D + 1}$

$= e^{-z} \dfrac{1}{(D-1)^2 - 4(D-1) + 1} z \sin z$

$= e^{-z} \left(\dfrac{1}{D^2 - 6D + 6} \right) z \sin z$

$\left[\because \dfrac{1}{f(D)} x \cdot V = \left[x - \dfrac{1}{f(D)} f'(D) \right] \dfrac{1}{f(D)} \cdot V \right]$

$\left[\because \dfrac{1}{f(D)} z \cdot V = \left[z - \dfrac{1}{f(D)} f'(D) \right] \dfrac{1}{f(D)} \cdot V \right]$

$= e^{-z} \left[z - \dfrac{(2D-6)}{D^2 - 6D + D} \right] \dfrac{1}{D^2 - 6D + 6} \sin z$

$= e^{-z} \left(z - \dfrac{(2D-6)}{D^2 - 6D + 6} \right) \dfrac{\sin z}{D^2 - 6D + 6} \quad (D^2 = -a^2) = -1^2 = -1.$

$= e^{-z} \left[z - \dfrac{(2D-6)}{D^2 - 6D + 6} \right] \dfrac{\sin z}{5 - 6D} \times \dfrac{5 + 6D}{5 + 6D} (D^2 = -1)$

$= e^{-z} \left[\dfrac{z (5 \sin z + 6 \cos z)}{61} - \dfrac{(2D-6)(5 \sin z + 6 \cos z)}{61 (5 - 6D)} \right]$

$y_{p_1} = e^{-z} \left[\log x [5 \sin(\log x) + 6 \cos \log x] + \dfrac{54 \sin(\log x) + 382 \cos(\log x)}{61} \right]$

$\therefore \boxed{y_p = y_{p_1} + y_{p_2}}$ ——— (4)

sub. (3) & (4) in $\boxed{y = y_c + y_p}$ //

Legendre's Equation :- An Eqⁿ of the form Legendre's L.D.E is of the form $(a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 (a+bx)^{n-2} \frac{d^{n-2} y}{dx^{n-2}}$

$$+ \cdots + a_n y = \phi(x) \quad \text{--- (1)}$$

where $a_1, a_2, \ldots, a_n$ are real constants and $\phi(x)$ is the function of 'x' is called a Legendre's L.D.E.

The operator form of Eqⁿ (1) is

$$\left[(a+bx)^n D^n + a_1 (a+bx)^{n-1} D^{n-1} + \cdots + a_n\right] y = \phi(x).$$

Eqⁿ (1) can be reduced into L.D.E by substituting the values $a+bx = e^z \rightarrow z = \log(a+bx)$.

$$\frac{dz}{dx} = \frac{1}{a+bx} \cdot b$$

Consider $\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \Rightarrow Dy = \theta y \frac{1}{a+bx} \cdot b,$

$$\Rightarrow (a+bx)^1 D = b\theta$$

$$\text{||}^{ly} \quad (a+bx)^2 D^2 = b^2 \theta(\theta-1)$$

$$\& \quad (a+bx)^3 D^3 = b^3 \theta(\theta-1)(\theta-2)$$

$$\vdots$$

$$(a+bx)^n D^n = b^n \left[\theta(\theta-1)(\theta-2)\cdots(\theta-(n-1))\right]$$

Q. Solve $(x+1)^2 \frac{d^2 y}{dx^2} - 3(x+1)\frac{dy}{dx} + 4y = x^2+x+1$

Solⁿ: Clearly the given D.E is Legendre's L.D.E.

The operator form is

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$$\left[(x+1)^2D^2-3(x+1)D+4\right]y=x^2+x+1 \quad \text{--- (1)}$$

let $x+1=e^z \rightarrow x=e^z-1.$

$\rightarrow z=\log(x+1)$

Take $(x+1)D=b.\theta=1\theta=\theta.$ $(\because b=1)$

$(x+1)^2D^2=b^2\theta(\theta-1)=\theta^2-\theta.$

sub. in (1)

$\rightarrow \left[(\theta^2-\theta)-3\theta+4\right]y=x^2+x+1$

$\Rightarrow (\theta^2-4\theta+4)y=x^2+x+1$

$\Rightarrow (\theta^2-4\theta+4)y=(e^z-1)^2+e^z-\cancel{1}+\cancel{1}$

$\rightarrow (\theta^2-4\theta+4)y=e^{2z}+1-e^z$

$\sim f(\theta)y=\phi(z)$

$\therefore \boxed{y=y_c+y_p}$ --- (2)

$y_c=(c_1+c_2z)e^{2z}$ --- (3)

$$y_p=\frac{1}{f(\theta)}\phi(z)=\frac{1}{\theta^2-4\theta+4}\left(e^{2z}-e^z+1\right)$$

$$=\frac{e^{2z}}{\theta^2-4\theta+4}-\frac{e^z}{\theta^2-4\theta+4}+\frac{1}{\theta^2-4\theta+4}$$

$$=\frac{e^{2z}}{(\theta-2)^2}-e^z+\frac{1}{4}$$

$$=\frac{e^{2z}\cdot z^2}{2!}-e^z+1/4$$

$$y_p=\frac{z^2\cdot e^{2z}}{2}-e^z+1/4. \quad \text{--- (4)}$$

Sub$_{11}$ (3) & (4) in (2).

$$y = y_c + y_p$$

$$\Rightarrow y = [c_1 + c_2(\log(x+1))](x+1)^2 + \frac{[\log(x+1)]^2(x+1)^2}{2} - (x+1) + 1/4$$

② . Solve $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = \sin 2[\log(1+x)]$

Sol:- $[(1+x)^2 D^2 + (1+x)D + 1]y = \sin 2\log(1+x)$ ——— (1)

let $x+1 = e^z \Rightarrow x = e^z - 1$

& $\log(x+1) = z$.

Take. $(1+x)D = b\theta = 1\theta = \theta \quad (\because b=1)$

$(1+x)^2 D^2 = b^2\theta(\theta-1) = \theta^2 - \theta$

$(1) \Rightarrow (\theta^2 - \theta + \theta + 1)y = \sin 2z$

$\Rightarrow (\theta^2+1)y = \sin 2z$

$f(\theta)y = \phi(z)$

$\boxed{y = y_c + y_p}$ ——— (2)

$\therefore y_c = e^{\alpha x}(c_1\cos z + c_2\sin z) = e^{0.x}(c_1\cos z + c_2\sin z)$

$\boxed{y_c = c_1\cos z + c_2\sin z}$ ——— (3).

$y_p = \frac{1}{\theta^2+1}\sin 2z \quad (\because \theta^2 = -a^2 = -2^2)$

$\boxed{y_p = \frac{\sin 2z}{-3}}$ ——— (4)

$\therefore \boxed{y = y_c + y_p}$

③ . $(2x-1)^3\frac{d^3y}{dx^3} + (2x-1)\frac{dy}{dx} - 2y = x.$

Sol:- $[(2x-1)^3 D^3 + (2x-1)D - 2]y = x$ ——— (1)

put $2x-1 = e^z \Rightarrow x = \frac{e^z+1}{2}$

& $\log(2x-1) = z$. $\quad b=2;\ a=-1$.

$(2x-1)D = b\theta = 2\theta \quad \left(\theta = \frac{d}{dz}\right)$

$(2x-1)^2 D^2 = b^2\theta(\theta-1) = 4\theta(\theta-1)$

$(2x-1)^3 D^3 = 8\theta(\theta-1)(\theta-2)$.

(1) $\Rightarrow (8\theta^3 - 24\theta^2 + 18\theta - 2)y = \frac{e^z+1}{2}$

$f(\theta)y = Q(z)$

$\therefore \boxed{y = y_c + y_p}$ —— (2)

$$\begin{array}{c|cccc} m=1 & 8 & -24 & 18 & -2 \\ & 0 & 8 & -16 & 2 \\ \hline & 8 & -16 & 2 & \boxed{0} \end{array}$$

$8m^2 - 16m + 2 = 0$

$\Rightarrow 4m^2 - 8m + 1 = 0$

$m = \frac{8 \pm \sqrt{64-16}}{8}$

$= \frac{8 \pm \sqrt{48}}{8}$

$= \frac{8 \pm 4\sqrt{3}}{8}$

$= 1 \pm \frac{\sqrt{3}}{2}$

$\underline{y_c}$:- $m_1 = 1$; $m = 1 \pm \frac{\sqrt{3}}{2}$

$\left(\alpha \pm \sqrt{3/4}\right)$

$\therefore y_c = (c_1 \cosh\sqrt{\beta}z + c_2 \sinh\sqrt{\beta}z)e^{\alpha z} + c_3 e^{m_1 z}$

$y_c = e^{z}\left[c_1 \cosh\sqrt{3/4}\,z + c_2 \sinh\sqrt{3/4}\,z\right] + c_3 e^{z}$. —— (3)

& $y_p = \frac{1}{f(\theta)} Q(z)$

$= \frac{1}{8\theta^3 - 24\theta^2 + 18\theta - 2} \frac{e^z}{2} + \frac{1}{2} \frac{1}{8\theta^3 - 24\theta^2 + 18\theta - 2} e^{0 \cdot z}$

(put $\theta = 0$)

$= \frac{1}{2}\left[\frac{e^z}{(\theta-1)(8\theta^2 - 16\theta + 2)}\right] + \frac{1}{2}\frac{1}{(-2)}$

$= \frac{1}{2} \frac{z^1 e^z}{1!\,(-6)}$ (put $\theta = 1$)

$y_p = \frac{-ze^z}{12} - 1/4$. —— (4)

$\therefore \boxed{y = y_c + y_p}$.

4. $(3x+2)^2\dfrac{d^2y}{dx^2}+3(3x+2)\cdot\dfrac{dy}{dx}-36y=3x^2+4x+1$ (19)

Sol: $\left[(3x+2)^2D^2+3(3x+2)D-36\right]y=3x^2+4x+1$ —— (1)

Put $3x+2=e^z \Rightarrow x=(e^z-2)/3$

$\Rightarrow \log(3x+2)=z$

$\left(a=2,\ b=3,\ \theta=\dfrac{d}{dz}\right)$

& $(3x+2)D=b\theta=3\theta$

$(3x+2)^2D^2=9\theta(\theta-1)$

$(1)\Rightarrow (9\theta^2-36)y=\dfrac{e^{2z}-1}{3}$

$f(\theta)y=Q(z)$

$\therefore \boxed{y=y_c+y_p}$ —— (2)

$y_c=c_1e^{-2z}+c_2e^{2z}$ —— (3)

$y_p=\dfrac{1}{9\theta^2-36}\left(\dfrac{e^{2z}-1}{3}\right)$

$=\dfrac{1}{3}\left(\dfrac{e^{2z}}{9\theta^2-36}-\dfrac{e^{0\cdot z}}{9\theta^2-36}\right)$

$=\dfrac{1}{3}\left[\dfrac{e^{2z}}{9\theta^2-36}+\dfrac{1}{36}\right]$

$=\dfrac{1}{3}\left[\dfrac{e^{2z}}{(3\theta+6)(3\theta-6)}+\dfrac{1}{36}\right]$

$=\dfrac{1}{3}\left(\dfrac{e^{2z}\,z}{36\ 1!}+\dfrac{1}{36}\right)$

$y_p=\dfrac{ze^{2z}}{108}+\dfrac{1}{108}$ —— (4)

$\boxed{\therefore y=y_c+y_p}$

(5). $x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + 2y = x\log x$

Sol: $(x^2D^2 - xD + 2)y = x\log x$

put $x = e^z \Rightarrow \log x = z$

$\&\ xD = \theta$

$x^2D^2 = \theta(\theta-1)$

$\Rightarrow (\theta^2 - \theta - \theta + 2)y = e^z z$

$\Rightarrow (\theta^2 - 2\theta + 2)\, y = ze^z$

$\therefore y_c = (c_1\cos z + c_2\sin z)\, e^z$

$m^2 - 2m + 2 = 0$

$m = \frac{-(-2) \pm \sqrt{4 - 4(2)}}{2}$

$= \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm \sqrt{4i^2}}{2} = \frac{2 \pm 2i}{2}$

$= 1 \pm i$

$y_p = \frac{1}{\theta^2 - 2\theta + 2}\, ze^z$

$= e^z \frac{1}{(\theta+1)^2 - 2(\theta+1) + 2} \cdot z$

$= e^z\left(\frac{1}{\theta^2+1}\right) z = e^z (1+\theta^2)^{-1} z$

$= e^z(1 - \theta^2.)\, z$

$= e^z(z)$

$= ze^z$

$\therefore \boxed{y = y_c + y_p}$

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Assignment Questions

(UNIT-II)

1. a) Solve $(D^2-2D+5)y = e^{2x}\sin x$

   b) Solve $y''-2y'+2y = e^{x}\tan x$ by using method of variation of parameters.

2. a) Solve $(D^3-D^2-D-2)y=0$

   b) Find particular integral of $(D-1)^4 y = e^{x}$

3. a) Solve $(D^2-2D+1)y = xe^{x}\sin x$

   b) Solve $y''-4y'+3y = 4e^{3x}$; $y(0)=-1$
   $y'(0)=3$.

4. a) Solve $(D^2-4)y = 2\cosh^2 x$.

   b) Solve $(D^2+1)y = x\cos 2x$ given $y(0)=0$ and
   $y'(0)=0$.

5. a) Solve $x^3D^3y + 2x^2D^2y + 2y = 10\left(x+\frac{1}{x}\right)$

   b) Solve $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1$

⑥. Solve $(x^2D^2+4xD+2)y = e^x$ — (1)

Sol^n The given Eq^n is in the operator form of Cauchy-Euler Eq^n.

$x = e^z \Rightarrow \log x = z$

$xD = \theta$

$x^2D^2 = \theta(\theta-1)$

(1) $\Rightarrow (\theta^2-\theta+4\theta+2)y = e^{e^z}$

$\Rightarrow (\theta^2+3\theta+2)y = e^{e^z}$

$f(\theta)y = \phi(z)$

$\boxed{y = y_c+y_p}$ — (2)

Here $y_c \Rightarrow m^2+3m+2=0$

$m = -2, -1.$

$\therefore y_c = c_1 e^{-2z} + c_2 e^{-z}$ — (3)

$y_p = \dfrac{1}{\theta^2+3\theta+2} e^{e^z}$

$= \left(\dfrac{1}{\theta+1} - \dfrac{1}{\theta+2}\right) e^{e^z}$ (by using partial fractions)

$= \dfrac{1}{\theta+1} e^{e^z} - \dfrac{1}{\theta+2} e^{e^z}$

$= e^{-z}\int e^{z} e^{e^z} dz - e^{-2z}\int e^{2z} e^{e^z} dz$ $\left(\because \dfrac{1}{D-a} f(x) = e^{ax}\int e^{-ax} f(x)\,dx + c\right.$

$\left.\because \dfrac{1}{D+a} f(x) = e^{-ax}\int e^{ax} f(x)\,dx + c\right)$

$= e^{-z}\int e^{t}\, dt - e^{-2z}\int e^{t}\, e^z dt$ $\left(\because \text{Put } e^z = t\right.$

$= e^{-z}\int e^t dt - e^{-2z}\int t e^t dt$ $\left. e^z dz = dt\right)$

$= e^{-z} e^{t} - e^{-2z} e^{t}(t-1) \quad \left[\because \int t e^{t}\, dt = e^{t}(t-1)\right]$

$= e^{-z} e^{e^{z}} - e^{-2z} e^{e^{z}} (e^{z}-1)$

$= e^{e^{z}} \left[e^{-z} - e^{-2z}(e^{z}-1) \right]$

$= e^{e^{z}} \left(\cancel{e^{-z}} - \cancel{e^{-z}} + e^{-2z} \right)$

$y_p = e^{e^{z}} e^{-2z}$ — (4)

$\therefore \boxed{y = y_c + y_p}$

7. $(D^2+1)y = 2^x$

Sol: $(m^2+1) = 0 \Rightarrow m^2 = -1$
$m = \pm i$

$\boxed{y_c = c_1 \cos x + c_2 \sin x}$

$y_p = \frac{1}{D^2+1} 2^x$

$= \frac{1}{D^2+1} e^{\log_e 2^x}$

$= \frac{1}{D^2+1} e^{x \log 2} \rightarrow a$ $\quad (\because \log a^x = x \log a)$

$= \frac{1}{D^2+1} e^{(\log 2)x}$ $\quad (e^{ax})$ where $a = \log 2$ [case (i)]

put $D = a = \log 2$

$= \frac{1}{(\log 2)^2+1} e^{(\log 2)x}$

$y_p = \frac{e^{(\log 2)x}}{1+(\log 2)^2}$

$\therefore y = y_c + y_p$